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ON THE SPECTRAL PROPERTIES OF SOME AUXILIARY BOUNDARY VALUE PROBLEMS FROM THEORY OF METAMATERIALS

We study new spectral boundary value problem arising in theory of metamaterials. We prove that spectrum of the general problem is discrete and situated in some sector. In the special one-dimensional case we find localization of eigenvalues tending to infinity and some asymptotic formulas.

Key words: Spectral boundary value problem, discrete spectrum, localization of eigenvalues, asymptotic formulas

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1. Introduction. The general statement of the problem

The present paper is devoted to study some model boundary value spectral problems arising in theory of composite materials. Such materials consist of many thin layers of materials with different properties that make possible to obtain some extra characteristics of composite material. Almost 40 years ago Russian scientist Victor Veselago had an idea for a material with negative index of refraction [1]. Such composite material with a lot of amazing attributes was obtained experimentally by the group of American physics at the beginning of XXI century (see [2], [3]). The negative refraction provides some effects that destroy the classical theory of electromagnetics. For example, superlens effect allows imaging of details finer than the wavelength of light used. Another possible application is effect of cloaking devices: at a given frequency, a spherical volume could be cloaked by means of a spherical shell within which the electric permittivity and magnetic permeability vary in certain prescribed ways. At the given frequency, any object contained within the spherical volume would be invisible to outside observers (see [4], [5]). Obviously, such phenomenons need for the physical and mathematical modeling.

We attempt to study some spectral boundary value problem generated by physical statements from [4], [5]. We consider the following general statement. One have to find the unknown functions $u_k(x)$ $(k = 1, 2, 3)$ (electromagnetic intensity) in given arbitrary smooth domains Ω_k that satisfy equations

$$-\alpha\Delta u_1(x) = \lambda u_1(x), \quad x \in \Omega_1, \tag{1}$$

$$-\beta\Delta u_2(x) = \lambda u_2(x), \quad x \in \Omega_2, \tag{2}$$

$$-\alpha\Delta u_3(x) = \lambda u_3(x), \quad x \in \Omega_3, \tag{3}$$

and boundary conditions

$$u_1(x) = u_2(x), \quad \alpha\frac{\partial u_1}{\partial n}(x) = \beta\frac{\partial u_2}{\partial n}(x), \quad x \in \Gamma_1, \tag{4}$$

$$u_2(x) = u_3(x), \quad \beta\frac{\partial u_2}{\partial n}(x) = \alpha\frac{\partial u_3}{\partial n}(x), \quad x \in \Gamma_2, \tag{5}$$

$$u_3(x) = 0, \qquad x \in \Gamma_3. \tag{6}$$

We suppose that domain Ω_1 situates inside Ω_2 including in Ω_3, and Γ_1 is common boundary of Ω_1 and Ω_2, Γ_2 is common boundary of Ω_2 and Ω_3 with outward normal vector $\vec{n}$. The requirement of unboundedness of Ω_3 we substitute for condition (6), where Γ_3 is sufficiently large outward boundary. The number $\lambda \in \mathbb{C}$ is unknown spectral parameter, complex numbers $\alpha,\ \beta \in \mathbb{C}$ $(0 \le \arg\alpha < \arg\beta \le \pi)$ are given as we consider. Notice that given statement corresponds to problems from [4], [5] for concentric rings Ω_k, where Ω_3 is unbounded and $\alpha = 1$, $\beta = -1 + i\varepsilon$ $(0 < \varepsilon << 1)$.

In the present work we introduce the first step to study the problem. We prove that problem (1)–(6) has the discrete spectrum with unique limit point at infinity that consist of isolated eigenvalues of finite multiplicity $\{\lambda_k\}_{k=1}^{\infty}$ situated in the sector $\arg\beta \le \arg\lambda \le \arg\alpha$. If domains Ω_k are one-dimension segments then for any given $\delta > 0$ there exists the number $R(\delta) > 0$ such that for $|\lambda| > R(\delta)$ we have no eigenvalues in the sector $\arg\alpha + \delta \le \arg\lambda \le \arg\beta - \delta$. In polar coordinate system we have a branch of eigenvalues tending asymptotically to some parabola with axis of symmetry $\varphi = \arg\beta$ and two branches of eigenvalues tending asymptotically to another parabola with axis of symmetry $\varphi = \arg\alpha$.

2. Properties of the problem in arbitrary domains Ω_k

Suppose that element $u = (u_1(x); u_2(x); u_3(x)) \in E := L_2(\Omega_1) \oplus L_2(\Omega_2) \oplus L_2(\Omega_3)$ is a solution of (1)–(6). Left-hand-sides of (1)–(3) determine the linear operator $\mathcal{A}$ in E. If we set it on the elements u with properties: $u_k(x) \in C^2(\Omega_k)$ $(k = 1, 2, 3)$ and $u_k(x)$ satisfy boundary conditions (4)–(6), then using first Green's formula we obtain

$$(\mathcal{A}u, u)_E = -\alpha\int\limits_{\Omega_1} \Delta u_1\overline{u_1}\, d\Omega_1 - \beta\int\limits_{\Omega_2} \Delta u_2\overline{u_2}\, d\Omega_2 - \alpha\int\limits_{\Omega_3} \Delta u_3\overline{u_3}\, d\Omega_3 =$$

$$= \alpha \int\limits_{\Omega_1} |\nabla u_1|^2 \, d\Omega_1 + \beta \int\limits_{\Omega_2} |\nabla u_2|^2 \, d\Omega_2 + \alpha \int\limits_{\Omega_3} |\nabla u_3|^2 \, d\Omega_3 -$$

$$- \alpha \int\limits_{\Gamma_1} \frac{\partial u_1}{\partial n} \overline{u_1}|_{\Gamma_1} \, d\Gamma_1 + \beta \int\limits_{\Gamma_1} \frac{\partial u_2}{\partial n} \overline{u_2}|_{\Gamma_1} \, d\Gamma_1 - \beta \int\limits_{\Gamma_2} \frac{\partial u_2}{\partial n} \overline{u_2}|_{\Gamma_2} \, d\Gamma_2 + \alpha \int\limits_{\Gamma_2} \frac{\partial u_3}{\partial n} \overline{u_3}|_{\Gamma_2} \, d\Gamma_2 -$$

$$- \beta \int\limits_{\Gamma_3} \frac{\partial u_3}{\partial n} \overline{u_3}|_{\Gamma_3} \, d\Gamma_3 = \alpha \int\limits_{\Omega_1} |\nabla u_1|^2 \, d\Omega_1 + \beta \int\limits_{\Omega_2} |\nabla u_2|^2 \, d\Omega_2 + \alpha \int\limits_{\Omega_3} |\nabla u_3|^2 \, d\Omega_3.$$

Therefore $(\mathcal{A}u, u)_E = c_1\alpha + c_2\beta$, where $c_1, c_2 \geq 0$. So, the numerical range of the operator $\mathcal{A}$ is the sector of complex plane, forming by polar axes $\varphi = \arg\alpha$ and $\varphi = \arg\beta$. In particular, the case $\alpha, \beta > 0$ corresponds to nonnegative operator. If $\alpha > 0$, $\beta < 0$ (superlens without dissipation of energy) then we have a symmetric operator.

To prove the discreteness of the spectrum let us change the spectral parameter λ to the parameter $\mu = \dfrac{\lambda}{\beta} + 1$. Then equations (1)–(3) can be rewritten as

$$-\Delta u_2(x) + u_2(x) = \mu u_2(x), \qquad x \in \Omega_2, \tag{7}$$

$$-\Delta u_k(x) + u_k(x) = \frac{\beta\mu - 1 + \alpha}{\beta} u_k(x), \quad x \in \Omega_k \quad (k = 1, 3). \tag{8}$$

The following Green's formulas are valid:

$$\alpha \int\limits_{\Omega_1} (-\Delta u_1 + u_1)\overline{\eta_1} \, d\Omega_1 = \alpha \int\limits_{\Omega_1} (\nabla u_1 \cdot \nabla \eta_1 + u_1\overline{\eta_1}) \, d\Omega_1 - \alpha \int\limits_{\Gamma_1} \frac{\partial u_1}{\partial n} \overline{\eta_1}|_{\Gamma_1} \, d\Gamma_1;$$

$$\beta \int\limits_{\Omega_1} (-\Delta u_2 + u_2)\overline{\eta_2} \, d\Omega_2 = \beta \int\limits_{\Omega_2} (\nabla u_2 \cdot \nabla \eta_2 + u_2\overline{\eta_2}) \, d\Omega_2 +$$

$$+ \beta \int\limits_{\Gamma_1} \frac{\partial u_2}{\partial n} \overline{\eta_2}|_{\Gamma_1} \, d\Gamma_1 - \beta \int\limits_{\Gamma_2} \frac{\partial u_2}{\partial n} \overline{\eta_2}|_{\Gamma_2} \, d\Gamma_2;$$

$$\alpha \int\limits_{\Omega_3} (-\Delta u_3 + u_3)\overline{\eta_3} \, d\Omega_3 = \alpha \int\limits_{\Omega_3} (\nabla u_3 \cdot \nabla \eta_3 + u_3\overline{\eta_3}) \, d\Omega_3 +$$

$$+ \alpha \int\limits_{\Gamma_2} \frac{\partial u_3}{\partial n} \overline{\eta_3}|_{\Gamma_2} \, d\Gamma_2 - \alpha \int\limits_{\Gamma_3} \frac{\partial u_3}{\partial n} \overline{\eta_3}|_{\Gamma_3} \, d\Gamma_3.$$

Suppose now that

$$\eta = (\eta_1; \eta_2; \eta_3) \in F_0 := \{\eta_k \in H^1(\Omega_k) : \ \eta_1|_{\Gamma_1} = \eta_2|_{\Gamma_1}, \ \eta_2|_{\Gamma_2} = \eta_3|_{\Gamma_2}, \ \eta_3|_{\Gamma_3} = 0\}$$

and the element $u = (u_1; u_2; u_3) \in E$ is a solution of spectral problem. Then we have the identity

$$(\beta\mu - 1 + \alpha) \int\limits_{\Omega_1} u_1\overline{\eta_1} \, d\Omega_1 + \beta\mu \int\limits_{\Omega_2} u_2\overline{\eta_2} \, d\Omega_2 + (\beta\mu - 1 + \alpha) \int\limits_{\Omega_3} u_3\overline{\eta_3} \, d\Omega_3 =$$

$$= \alpha(u_1, \eta_1)_{H^1(\Omega_1)} + \beta(u_2, \eta_2)_{H^1(\Omega_2)} + \alpha(u_3, \eta_3)_{H^1(\Omega_3)}.$$

As the spaces $H^1(\Omega_k)$ (with the standard inner products) are boundedly (compactly) imbedded into $L_2(\Omega_k)$ $(k = 1, 3)$, there exist positive compact in $L_2(\Omega_k)$ operators V_k which satisfy the identities

$$\int_{\Omega_k} u_k \overline{\eta_k}\, d\Omega_k = (V_k u_k, \eta_k)_{H_1(\Omega_k)}.$$

On the base of these formulas we have

$$\beta\mu(u, \eta)_E = (\alpha u_1 + (1-\alpha)V_1 u_1, \eta_1)_{H^1(\Omega_1)} + \beta(u_2, \eta_2)_{H^1(\Omega_2)} + (\alpha u_3 + (1-\alpha)V_3, \eta_3)_{H^1(\Omega_3)}.$$

Hence

$$\mu(u, \eta)_E = (\mathcal{B}u, \eta)_{F_0},$$

where $\mathcal{B} = \mathrm{diag}\,\{\frac{\alpha}{\beta} I_1 + (1-\alpha)V_1;\ I_2;\ \frac{\alpha}{\beta} I_3 + (1-\alpha)V_3\}$ in E. As the space F_0 is compactly imbedded into E (see, for e.g. [6]), there exists positive definite in E operator $\mathcal{A}_0$ such that $(\mathcal{B}u, \eta)_{F_0} = (\mathcal{A}_0 \mathcal{B}u, \eta)_E$ for any element $\eta \in F_0$. Therefore μ is an eigenvalue of unbounded in E operator $\mathcal{A}_0\mathcal{B}$.

It is easy to show that if $\frac{\beta}{\alpha} - \beta \notin \mathbb{R}_-$ then the operator $\mathcal{B}$ is boundedly invertible. In this case the numbers μ are characteristic numbers of compact operator $\mathcal{B}^{-1}\mathcal{A}_0^{-1}$. Therefore they form the countable set of values with unique limit point at infinity. The same property is valid for the eigenvalues $\lambda = \beta\mu - 1$.

Let us consider separately the case $\alpha = 1$. We obtain $\mathcal{B}^{-1} = \mathrm{diag}\,\{\beta I_1;\ I_2;\ \beta I_3\}$. If additionally $\beta = 1$ then $\mathcal{B}^{-1} = \mathcal{I}$. Therefore the spectrum of the problem is the set of positive eigenvalues of the operator $\mathcal{A}_0$. It tends to infinity, and the set of eigenfunction form the orthonormal basis in E (analogous result in the space E equipped with equivalent inner product is valid for $\beta > 0$).

If $\beta = -1$ then the operator $\mathcal{B}^{-1} = \mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$ is the operator of canonical symmetry in E, and the operator $\mathcal{B}^{-1}\mathcal{A}_0^{-1}$ is $\mathcal{J}$-positive in the space of M. Krein with the indefinite inner product $[u, \eta] = (\mathcal{J}u, \eta)_E$. One can prove that the operator $\mathcal{J}\mathcal{A}_0^{-1}$ has infinite dimensional positive and negative maximal invariant subspaces $L_\pm$. So, in these spaces the operator has $\mathcal{J}-$orthonormal systems of eigenelements $\{e_k^\pm\}_{k=1}^\infty$, $[e_k^\pm, e_k^\pm] = \pm 1$, with corresponding branches of positive and negative eigenvalues of the operator $\mathcal{J}\mathcal{A}_0^{-1}$ with a unique limit point at zero (see, e.g., [7], p. 41-42). Therefore the problem has branches of positive and negative eigenvalues with limit points at $\pm\infty$.

The problem with $\beta \notin \mathbb{R}$ is rather difficult. To show it we can write the matrix form of the operator

$$\mathcal{A}_0\mathcal{B} = \begin{pmatrix} \beta^{-1}A_{11} & \beta^{-1}A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & \beta^{-1}A_{32} & \beta^{-1}A_{33} \end{pmatrix}.$$

The spectrum of the boundary value problem (1)–(6) coincides with the spectrum of this operator. Here we have unbounded operator coefficients A_{ij} without any subordinates, so the classical approaches are not effective to study the spectral properties.

3. Characteristic equation in one-dimensional case

Let $\arg\alpha < \arg\beta$ and $\Omega_1 = (-R;-l)$, $\Omega_2 = (-l;l)$, $\Omega_3 = (l;R)$ be three intervals then problem (1)–(6) can be rewritten in the form

$$-\alpha u_1''(x) = \lambda u_1(x), \qquad x \in (-R;-l), \tag{9}$$

$$-\beta u_2''(x) = \lambda u_2(x), \qquad x \in (-l;l), \tag{10}$$

$$-\alpha u_3''(x) = \lambda u_3(x), \qquad x \in (l;R), \tag{11}$$

$$u_1(0) = u_2(0), \qquad u_2(l) = u_3(l), \tag{12}$$

$$\alpha u_1'(-l) = \beta u_2'(-l), \qquad \beta u_2'(l) = \alpha u_3'(l), \tag{13}$$

$$u_1(-R) = u_3(R) = 0. \tag{14}$$

Let us consider two auxiliary problems

$$\begin{cases} -u_1''(x) = \dfrac{\lambda}{\alpha}\, u_1(x), \quad x \in (-R;-l), \\ u_1(-R) = 0, \end{cases} \qquad \begin{cases} -u_3''(x) = \dfrac{\lambda}{\alpha}\, u_3(x), \quad x \in (l;R), \\ u_3(R) = 0. \end{cases} \tag{15}$$

Their solutions are

$$u_1(x) = d_1 \sin\sqrt{\frac{\lambda}{\alpha}}(x+R), \qquad u_3(x) = d_3 \sin\sqrt{\frac{\lambda}{\alpha}}(x-R). \tag{16}$$

Therefore

$$\frac{u_1'(-l)}{u_1(-l)} = \sqrt{\frac{\lambda}{\alpha}} \operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) =: m(\lambda), \qquad \frac{u_3'(l)}{u_3(l)} = -\sqrt{\frac{\lambda}{\alpha}} \operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) = -m(\lambda). \tag{17}$$

Using this notations we obtain the problem for $u_2(x)$:

$$-u_2''(x) = \frac{\lambda}{w} u_2(x), \quad x \in (0;l), \tag{18}$$

$$\beta u_2'(-l) = \alpha u_1'(-l) = \alpha m_1(\lambda) u_1(-l) = \alpha m(\lambda) u_2(-l), \tag{19}$$

$$\beta u_2'(l) = \alpha u_3'(l) = \alpha m_3(\lambda) u_3(l) = -\alpha m(\lambda) u_3(l). \tag{20}$$

Equation (18) has the solution $u_2(x) = c_1 \sin\sqrt{\frac{\lambda}{\beta}}x + c_2 \cos\sqrt{\frac{\lambda}{\beta}}x$. So, boundary conditions (19)–(20) imply

$$\beta u_2'(-l) = \beta\sqrt{\frac{\lambda}{\beta}}\left[c_1 \cos\sqrt{\frac{\lambda}{\beta}}l + c_2 \sin\sqrt{\frac{\lambda}{\beta}}l\right] = \alpha m(\lambda)\left[-c_1 \sin\sqrt{\frac{\lambda}{\beta}}l + c_2 \cos\sqrt{\frac{\lambda}{\beta}}l\right],$$

$$\beta u_2'(l) = \beta\sqrt{\frac{\lambda}{\beta}}\left[c_1 \cos\sqrt{\frac{\lambda}{\beta}}l - c_2 \sin\sqrt{\frac{\lambda}{\beta}}l\right] = -\alpha m(\lambda)\left[c_1 \sin\sqrt{\frac{\lambda}{\beta}}l + c_2 \cos\sqrt{\frac{\lambda}{\beta}}l\right].$$

It is linear homogeneous system of two equation with unknown constants c_1 and c_2 which has nontrivial solutions if and only if

$$\det = \begin{vmatrix} \sqrt{\beta\lambda}\cos\sqrt{\frac{\lambda}{\beta}}l + \alpha m(\lambda)\sin\sqrt{\frac{\lambda}{\beta}}l & \sqrt{\beta\lambda}\sin\sqrt{\frac{\lambda}{\beta}}l - \alpha m(\lambda)\cos\sqrt{\frac{\lambda}{\beta}}l \\ \sqrt{\beta\lambda}\cos\sqrt{\frac{\lambda}{\beta}}l + \alpha m(\lambda)\sin\sqrt{\frac{\lambda}{\beta}}l & -\sqrt{\beta\lambda}\sin\sqrt{\frac{\lambda}{\beta}}l + \alpha m(\lambda)\cos\sqrt{\frac{\lambda}{\beta}}l \end{vmatrix} = 0. \quad (21)$$

Hence

$$2\left(\sqrt{\beta\lambda}\cos\sqrt{\frac{\lambda}{\beta}}l + \alpha m(\lambda)\sin\sqrt{\frac{\lambda}{\beta}}l\right)\left(\sqrt{\beta\lambda}\sin\sqrt{\frac{\lambda}{\beta}}l - \alpha m(\lambda)\cos\sqrt{\frac{\lambda}{\beta}}l\right) = 0. \quad (22)$$

If $\sin\sqrt{\frac{\lambda}{\beta}}l = 0$ then (22) implies that either $\lambda = 0$ or $m(\lambda) = 0$. One can check that $\lambda = 0$ is not eigenvalue. Therefore

$$m(\lambda) = \sqrt{\frac{\lambda}{\alpha}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) = 0.$$

So, we obtain $\sqrt{\lambda} = \frac{\sqrt{\alpha}}{R-l}(\frac{\pi}{2} + \pi n)$ $(n \in \mathbb{Z})$. On the other hand by the assumption we have $\sqrt{\lambda} = \frac{\sqrt{\beta}}{l}\pi k$ $(k \in \mathbb{Z})$. It can not be possible since $\arg\alpha \neq \arg\beta$, hence $\sin\sqrt{\frac{\lambda}{\beta}}l \neq 0$.

Equation (22) implies two characteristic equations

$$f_1(\lambda) := \operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l + \sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) = 0, \quad (23)$$

$$f_2(\lambda) := -\operatorname{tg}\sqrt{\frac{\lambda}{\beta}}l + \sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) = 0. \quad (24)$$

These equations can be reduced to the problem of zeros of some entire functions, so the spectrum is discrete and it has the unique limit point at infinity.

4. Localization of the eigenvalues with great absolute value

Let us study the case $|\lambda| \to \infty$. To this aim we divide the numerical range of the operator $\mathcal{A}$ to the three domains $V_\alpha := \{z \in \mathbb{C} : \arg\alpha < \arg z < \arg\alpha + \delta\}$, $V_0 := \{z \in \mathbb{C} : \arg\alpha + \delta < \arg z < \arg\beta - \delta\}$, $V_\beta := \{z \in \mathbb{C} : \arg\beta - \delta < \arg z < \arg\beta\}$, where $\delta > 0$ is a given little number.

For $|\lambda| \to \infty$ inside the domain $V_0 \cup V_\alpha$ we have

$$\operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l \to i. \quad (25)$$

Analogously, inside the domain $V_0 \cup V_\beta$ we have

$$\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) \to -i. \quad (26)$$

Lemma 1. *There exist such a number $R(\delta) > 0$ that there is no solutions of* (23), (24) *in the domain V_0 for $|\lambda| > R(\delta)$.*

Proof. Indeed, let us suppose that we have infinitely many solutions $\{\lambda_{n_k}\}$ of (23) situated in the V_0. Using (25), (26) we obtain

$$f_1(\lambda_{n_k}) \to i - \sqrt{\frac{\alpha}{\beta}}\, i = i\frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\beta}} \neq 0 \quad (n \to \infty), \tag{27}$$

since $\arg\alpha < \arg\beta$. It is impossible as $f(\lambda_{n_k}) \equiv 0$. Analogous result is obviously valid for (24). □

Using asymptotic (25) inside the domain $V_0 \cup V_\alpha$ we obtain that (23) and (24) are close to equation

$$\widetilde{f}_1(\lambda) := i + \sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) = 0. \tag{28}$$

Inside the domain $V_0 \cup V_\beta$ we have asymptotic (26), therefore equations (23) and (24) are close to

$$\widetilde{f}_{21}(\lambda) := \operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l - \sqrt{\frac{\alpha}{\beta}}i = 0, \tag{29}$$

$$\widetilde{f}_{22}(\lambda) := \operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l - \sqrt{\frac{\beta}{\alpha}}i = 0. \tag{30}$$

Union (29), (30) can be reduced to the unique equation. Suppose that

$$\widetilde{f}_{21}(\lambda)\cdot\widetilde{f}_{22}(\lambda) = \operatorname{ctg}^2\sqrt{\frac{\lambda}{\beta}}l - \operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l\left(\sqrt{\frac{\alpha}{\beta}}i + \sqrt{\frac{\beta}{\alpha}}i\right) - 1 =$$

$$= \frac{\cos^2\sqrt{\frac{\lambda}{\beta}}l - \sin^2\sqrt{\frac{\lambda}{\beta}}l}{\sin^2\sqrt{\frac{\lambda}{\beta}}l} - \frac{2\cos\sqrt{\frac{\lambda}{\beta}}l\sin\sqrt{\frac{\lambda}{\beta}}l}{\sin^2\sqrt{\frac{\lambda}{\beta}}l}\left(\frac{i(\alpha+\beta)}{2\sqrt{\alpha\beta}}\right) = 0. \tag{31}$$

It is equivalent to

$$\operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}2l = \frac{i(\alpha+\beta)}{2\sqrt{\alpha\beta}}. \tag{32}$$

All of limit equations (28), (32) can be reduced to the form

$$\operatorname{ctg} z = a + ib, \qquad a, b \in \mathbb{R}. \tag{33}$$

Using the exponent we have $\dfrac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}} = a + ib$ or $i(e^{2iz}+1) = (a+ib)(e^{2iz}-1)$. Hence $e^{2iz} = \dfrac{a+i(b+1)}{a+i(b-1)}$. Finding the logarithm we obtain the countable set of solutions

$$z_n = \frac{1}{2}\arg\frac{a+i(b+1)}{a+i(b-1)} + \pi n - \frac{i}{2}\ln\left|\frac{a+i(b+1)}{a+i(b-1)}\right| =: A + \pi n - iB, \quad n \in \mathbb{Z}\ (A, B \in \mathbb{R}). \tag{34}$$

Therefore corresponding solutions of (28) have the form

$$\lambda_n = \frac{\alpha z^2}{(R-l)^2} = \frac{\alpha}{|\alpha|}\Big[\frac{|\alpha|((A_1+\pi n)^2 - B_1^2)}{(R-l)^2} - 2i\frac{|\alpha| B_1(A_1+\pi n)}{(R-l)^2}\Big], \quad n \in \mathbb{N}. \tag{35}$$

All these points are situated on the branch of parabola

$$\operatorname{Re}\lambda = \frac{(\operatorname{Im}\lambda)^2(R-l)^2}{4|\alpha|B_1^2} - \frac{|\alpha|B_1^2}{(R-l)^2}, \tag{36}$$

rotated through the angle $\arg\alpha$. Notice that there exist some number N_1 such that for $n \geq N_1$ these points are situated in V_α. For $R - l \to \infty$ parabola (36) stretches along its axis of symmetry and coincides with the ray $\varphi = \arg\alpha$ in limit. This conclusion argues the hypothesis that the problem with unbounded domains has the continuous spectrum and it fills all the polar axis $\varphi = \arg\alpha$. This problem will be discussed in the further articles.

Analogous solutions of (32) have the form

$$\lambda_n = \frac{\beta z^2}{l^2} = \frac{\beta}{|\beta|}\Big[\frac{|\beta|((A_2+\pi n)^2 - B_2^2)}{4l^2} - 2i\frac{|\beta| B_2(A_2+\pi n)}{4l^2}\Big], \quad n \in \mathbb{N}. \tag{37}$$

All these points are situated on the branch of parabola

$$\operatorname{Re}\lambda = \frac{(\operatorname{Im}\lambda)^2 l^2}{|\beta|B_2^2} - \frac{|\beta|B_2^2}{4l^2}, \tag{38}$$

rotated through the angle $\arg\beta$. Obviously, there exist some number N_2 such that $\lambda_n \in V_\beta$ for $n \geq N_2$.

All of the limit equations are close to solutions of characteristic equations (23), (24) for $\lambda \to \infty$. More precise, for sufficiently large n solutions of characteristic and limit equations are as close as greater the number n.

Lemma 2. *For given $\gamma > 1$ and the sequence of positive numbers $r_n = n^{-\gamma} \to 0$ ($n \to \infty$) there exists such a number $N_\gamma \geq N_1 > 0$ that for all $n \geq N_\gamma$ in the neighborhood $|\lambda - \lambda_n| < r_n$ of each solution λ_n of* (28) *we have a unique solution of* (23) *and a unique solution of* (24).

Proof. Let λ_n ($n \geq N_1$) are solutions of (28) situated inside the domain V_α. To prove the lemma we can use the theorem of Rouche (see, e.g., [8], p. 131). It is sufficient to prove the inequality

$$|f_1(\lambda)f_2(\lambda) - \widetilde{f}_1^2(\lambda)| < |\widetilde{f}_1^2(\lambda)| \tag{39}$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_n| = r_n = n^{-\gamma}$ ($n \geq N_\gamma \geq N_1$).

Indeed, function $\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)$ is analytic inside V_α, so near λ_n we have

$$\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l) = \operatorname{ctg}\sqrt{\frac{\lambda_n}{\alpha}}(R-l) - \frac{(R-l)(\lambda-\lambda_n)}{2\sqrt{\alpha\lambda_n}\cdot\sin^2\sqrt{\frac{\lambda_n}{\alpha}}(R-l)} + O(|\lambda-\lambda_n|^2). \tag{40}$$

Using the assumption $\widetilde{f}_1(\lambda_n)=0$ we obtain

$$|\widetilde{f}_1^2(\lambda)| = |i+\sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)|^2 =$$

$$= |i+\sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda_n}{\alpha}}(R-l) - \frac{(R-l)(\lambda-\lambda_n)}{2\sqrt{\beta\lambda_n}\cdot\sin^2\sqrt{\frac{\lambda_n}{\alpha}}(R-l)} + O(|\lambda-\lambda_n|^2)|^2 >$$

$$> \left(\frac{(R-l)|\lambda-\lambda_n|}{|2\sqrt{\beta\lambda_n}\cdot\sin^2\sqrt{\frac{\lambda_n}{\beta}}(R-l)|} - |O(|\lambda-\lambda_n|^2)|\right)^2 = \left(\frac{\mathrm{const}\cdot r_n}{\sqrt{(A+\pi n)^2+B^2}} - O(r_n^2)\right)^2 =$$

$$= c\cdot n^{-2\gamma-2} + o(n^{-2\gamma-2}) \quad (n\to\infty,\ \gamma>1). \quad (41)$$

Here $|\sin^2\sqrt{\frac{\lambda_n}{\alpha}}(R-l)|>0$ is a fixed number, $|\sqrt{\lambda_n}| = c\cdot\sqrt{(A+\pi n)^2+B^2}$ according to (34), (35).

On the other hand

$$|f_1(\lambda)f_2(\lambda) - \widetilde{f}_1^2(\lambda)| =$$

$$= |\left(\operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l + \sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)\right)\left(-\operatorname{tg}\sqrt{\frac{\lambda}{\beta}}l + \sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)\right) -$$

$$- \left(i+\sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)\right)^2| = |\sqrt{\frac{\alpha}{\beta}}\operatorname{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)|\cdot|\operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l - \operatorname{tg}\sqrt{\frac{\lambda}{\beta}}l - 2i| <$$

$$< \left(\mathrm{const} + \frac{\mathrm{const}\cdot r_n}{\sqrt{(A+\pi n)^2+B^2}} + O(r_n^2)\right)\left(|\operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l - i| + |\operatorname{tg}\sqrt{\frac{\lambda}{\beta}}l + i|\right). \quad (42)$$

We have

$$|\operatorname{ctg}\sqrt{\frac{\lambda}{\beta}}l - i| = |i\frac{e^{2il\sqrt{\frac{\lambda}{\beta}}}+1}{e^{2il\sqrt{\frac{\lambda}{\beta}}}-1} - i| = \frac{2}{|e^{2il\sqrt{\frac{\lambda}{\beta}}}-1|} < \frac{2}{||e^{2il\sqrt{\frac{\lambda}{\beta}}}|-1|}. \quad (43)$$

For $\lambda\in V_\alpha$ we have $\operatorname{Im}\sqrt{\frac{\lambda}{\beta}}<0$. By the assumption $|\lambda-\lambda_n| = r_n\to 0$ and $\lambda\in V_\alpha$, so $\operatorname{Im}\sqrt{\frac{\lambda}{\beta}} < \operatorname{Im}\sqrt{\frac{\lambda_n}{\alpha}} + \varepsilon_n < 0$ for some $0<\varepsilon_n\to 0$ $(n\to\infty)$. If $\sqrt{\frac{\alpha}{\beta}} = a-ib$ then $\arg\beta > \arg\alpha$ imply $b>0$. So, using (34), (35) we obtain

$$\operatorname{Im}\sqrt{\frac{\lambda_n}{\beta}} = \operatorname{Im}\left(\sqrt{\frac{\lambda_n}{\alpha}}\cdot\sqrt{\frac{\alpha}{\beta}}\right) = \operatorname{Im}\left(c(A_1+\pi n+iB_1)(a-ib)\right) = -c(A_1+\pi n)b + caB_1, \quad c>0. \quad (44)$$

Therefore

$$|\exp(2il\sqrt{\frac{\lambda}{\beta}})| = \exp(-2l\operatorname{Im}\sqrt{\frac{\lambda}{\beta}}) > \exp(-2l\operatorname{Im}\sqrt{\frac{\lambda_n}{\beta}} - 2l\varepsilon_n) =$$

$$= \exp(-2l(-cb\pi n - cbA_1 + caB + \varepsilon_n)) > c_1 e^{c_2 n}, \quad c_1,\ c_2 > 0. \quad (45)$$

So, for $|\lambda - \lambda_n| = r_n \to 0$ there exist constants $c_1,\ c_2 > 0$ not depending on the number n such that

$$|\mathrm{ctg}\sqrt{\frac{\lambda}{\beta}}l - i| < \frac{2}{c_1 e^{c_2 n} - 1}, \quad (46)$$

For the same λ we have

$$|\mathrm{tg}\sqrt{\frac{\lambda}{\beta}}l + i| = |-i\frac{e^{2il\sqrt{\frac{\lambda}{\beta}}} - 1}{e^{2il\sqrt{\frac{\lambda}{\beta}}} + 1} + i| = \frac{2}{|e^{2il\sqrt{\frac{\lambda}{\beta}}} + 1|} < \frac{2}{|e^{2il\sqrt{\frac{\lambda}{\beta}}}| + 1} < \frac{2}{c_1 e^{c_2 n} + 1}. \quad (47)$$

As exponent increases to infinity faster then any power function we really can find such a number N_γ that

$$|f_1(\lambda)f_2(\lambda) - \widetilde{f}_1^2(\lambda)| < \mathrm{const}\cdot\left(\frac{2}{c_1 e^{c_2 n} - 1} + \frac{2}{c_1 e^{c_2 n} + 1}\right) < c\cdot n^{-2\gamma-2} + + o(n^{-2\gamma-2}) < |\widetilde{f}_1^2(\lambda)| \quad (48)$$

for $n \geq N_\gamma$ and $|\lambda - \lambda_n| = r_n = n^{-\gamma}$. □

Lemma 3. *For given $\gamma > 1$ and the sequence of positive numbers $r_n = n^{-\gamma} \to 0$ $(n \to \infty)$ there exist such a number $M_\gamma \geq N_2 > 0$ such that for all $n \geq M_\gamma$ in the neighborhood $|\lambda - \lambda_n| < r_n$ of each solution λ_n of* (29) *we have a unique solution of* (23) *and in the neighborhood $|\lambda - \lambda_n| < r_n$ of each solution λ_n of* (30) *we have a unique solution of* (24).

Proof. We prove only the first part of the theorem connected with the equations (29) and (23). The second part can be proved analogously.

Let λ_n $(n \geq N)$ are solutions of (29) situated inside the domain V_β. By the theorem of Rouche (see, e.g., [8], p. 131) it is sufficient to prove the inequality

$$|f_1(\lambda) - \widetilde{f}_{21}(\lambda)| = |\sqrt{\frac{\alpha}{\beta}}\,\mathrm{ctg}\sqrt{\frac{\lambda}{\alpha}}(R - l) + \sqrt{\frac{\alpha}{\beta}}i| < |\mathrm{ctg}\sqrt{\frac{\lambda}{\beta}}l - \sqrt{\frac{\alpha}{\beta}}i| = |\widetilde{f}_{21}(\lambda)| \quad (49)$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_n| = r_n = n^{-\gamma}$ and $n \geq M_\gamma \geq N_2$.

Indeed, function $\mathrm{ctg}\sqrt{\frac{\lambda}{\beta}}l$ is analytic inside V_β, and we have

$$\mathrm{ctg}\sqrt{\frac{\lambda}{\beta}}l = \mathrm{ctg}\sqrt{\frac{\lambda_n}{\beta}}l - \frac{l(\lambda - \lambda_n)}{2\sqrt{\beta\lambda_n}\cdot\sin^2\sqrt{\frac{\lambda_n}{\beta}}l} + O(|\lambda - \lambda_n|^2). \quad (50)$$

Using the assumption $\widetilde{f}_{21}(\lambda_n) = 0$ we obtain

$$|\widetilde{f}_{21}(\lambda)| = |-\frac{l(\lambda - \lambda_n)}{2\sqrt{\beta\lambda_n}\cdot\sin^2\sqrt{\frac{\lambda_n}{\beta}}l} + O(|\lambda - \lambda_n|^2)| >$$

$$> |\frac{i\sqrt{\alpha}l}{\sqrt{\lambda_n}\cdot\sin^2\sqrt{\frac{\lambda_n}{\beta}}l}|\cdot|\lambda-\lambda_n| - |O(|\lambda-\lambda_n|^2)| = \frac{\mathrm{const}\cdot r_n}{\sqrt{(A+\pi n)^2+B^2}} - O(r_n^2) =$$

$$= cn^{-\gamma-1} + O(n^{-2\gamma}) = cn^{-\gamma-1} + o(n^{-\gamma-1}) \quad (n\to\infty,\ \gamma>1). \quad (51)$$

Here $|\sin^2\sqrt{\frac{\lambda_n}{\beta}}l| > 0$ is a fixed number, $|\sqrt{\lambda_n}| = c\cdot\sqrt{(A+\pi n)^2+B^2}$ according to (34), (35).

On the other hand

$$|\mathrm{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)+i| = |i\frac{e^{2i(R-l)\sqrt{\frac{\lambda}{\alpha}}}+1}{e^{2i(R-l)\sqrt{\frac{\lambda}{\alpha}}}-1}+i| = \frac{2}{|1-e^{-2i(R-l)\sqrt{\frac{\lambda}{\alpha}}}|} < \frac{2}{|e^{-2i(R-l)\sqrt{\frac{\lambda}{\alpha}}}|-1}. \quad (52)$$

For all $\lambda_n \in V_\beta$ we have $\mathrm{Im}\sqrt{\frac{\lambda_n}{\alpha}} > 0$. By the assumption $|\lambda-\lambda_n| = r_n = n^{-\gamma}\to 0$ and $\lambda\in V_\beta$, so $\mathrm{Im}\sqrt{\frac{\lambda}{\alpha}} > \mathrm{Im}\sqrt{\frac{\lambda_n}{\alpha}} - \varepsilon_n > 0$ for some $0<\varepsilon_n\to 0$ $(n\to\infty)$. If $\sqrt{\frac{\beta}{\alpha}} = a+ib$ then $\arg\beta > \arg\alpha$ imply $b>0$. So, using (34), (35) we obtain

$$\mathrm{Im}\sqrt{\frac{\lambda_n}{\alpha}} = \mathrm{Im}\left(\sqrt{\frac{\lambda_n}{\beta}}\cdot\sqrt{\frac{\beta}{\alpha}}\right) = \mathrm{Im}\,(c(A+\pi n+iB)(a+ib)) = c(A+\pi n)b + caB, \quad c>0. \quad (53)$$

Therefore

$$|\exp(-2i(R-l)\sqrt{\frac{\lambda}{\alpha}})| = \exp(2(R-l)\mathrm{Im}\sqrt{\frac{\lambda}{\alpha}}) > \exp(2(R-l)\mathrm{Im}\sqrt{\frac{\lambda_n}{\alpha}} - 2R\varepsilon_n) =$$

$$= \exp(2(R-l)(cb\pi n + cbA + caB - \varepsilon_n)) > d_1 e^{d_2 n}. \quad (54)$$

for some $d_1,\ d_2 > 0$. Finally, we can find such a number M_γ that

$$|\widetilde{f}_{21}(\lambda)| = |\sqrt{\frac{\alpha}{\beta}}||\mathrm{ctg}\sqrt{\frac{\lambda}{\alpha}}(R-l)+i| < \frac{2}{d_1 e^{d_2 n}-1} < cn^{-\gamma-1} + o(n^{-\gamma-1}) < |\widetilde{f}_{21}(\lambda)|, \quad (55)$$

for $|\lambda-\lambda_n| = r_n = n^{-\gamma}$ and $n\ge M_\gamma$. □

5. Some asymptotic formulas for the limit equations

Now, let us consider the interesting special case

$$\alpha = 1, \quad \sqrt{\beta} = i+\varepsilon \quad (\beta = -1+\varepsilon^2+2i\varepsilon), \quad \varepsilon\to 0. \quad (56)$$

We can rewrite the limit equations (28), (32)

$$\mathrm{ctg}\left(\sqrt{\lambda}(R-l)\right) = 1 - i\varepsilon, \quad (57)$$

$$\mathrm{ctg}\left(\sqrt{\lambda}\frac{2l}{i+\varepsilon}\right) = -\frac{\varepsilon^2}{2(\varepsilon^2+1)} + i\frac{2\varepsilon+\varepsilon^3}{2(\varepsilon^2+1)} \approx -\frac{\varepsilon^2}{2} + i\varepsilon, \quad (58)$$

If $z = \sqrt{\lambda}(R-l)$ then we obtain by the formula (34)

$$z_n = \frac{\pi}{4} + \pi n - \frac{\varepsilon^2}{4} + i\left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{2}\right) + o(\varepsilon^2), \quad \varepsilon \to 0 \quad (n = 0, 1, 2, \dots). \tag{59}$$

Let us define $u_n := \frac{\pi}{4} + \pi n$, then we have asymptotic formula for the solutions of (57)

$$\lambda_n(\varepsilon) = \frac{z_n^2}{(R-l)^2} = \frac{1}{(R-l)^2}\left[u_n^2 - \varepsilon^2\frac{2u_n+1}{4} + i(\varepsilon u_n - \varepsilon^2 u_n)\right] + o(\varepsilon^2). \tag{60}$$

For $\varepsilon = 0$ we have the branch of positive eigenvalues

$$\lambda_n(0) = \frac{\pi^2(\frac{1}{4}+n)^2}{(R-l)^2}. \tag{61}$$

If ε increases then all this values move asymptotically in upper complex half plane along some parabolas, and its shift is as big as the number n. By formula (36) for all fixed $\varepsilon > 0$ the branch of eigenvalues is situated along the parabola

$$\operatorname{Re}\lambda = \frac{(\operatorname{Im}\lambda)^2(R-l)^2}{(\varepsilon-\varepsilon^2)^2} - \frac{(\varepsilon-\varepsilon^2)^2}{4(R-l)^2}. \tag{62}$$

If we change the variable $z = \dfrac{2l\sqrt{\lambda}}{i+\varepsilon}$ in (58) then using formula (34) we obtain

$$z_n = \frac{\pi}{2} + \pi n - \frac{\varepsilon^2}{2} - i(\varepsilon + 2\varepsilon^2) + o(\varepsilon^2), \quad \varepsilon \to 0 \quad (n = 0, 1, 2, \dots). \tag{63}$$

Let us define $v_n := \frac{\pi}{2} + \pi n$. So, we have asymptotic formula for the solutions of (58)

$$\lambda_n(\varepsilon) = \frac{-1+\varepsilon^2+2i\varepsilon}{4l^2} z_n^2 = \\ = \frac{1}{4l^2}\left[-v_n^2 + \varepsilon^2(1+5v_n+v_n^2) + 2i(\varepsilon(v_n+v_n^2) + 2\varepsilon^2 v_n)\right] + o(\varepsilon^2), \quad \varepsilon \to 0. \tag{64}$$

If $\varepsilon = 0$ then it is the branch of negative eigenvalues

$$\lambda_n(0) = -\frac{\pi^2(\frac{1}{2}+n)^2}{4l^2}. \tag{65}$$

If ε increases then all this values move asymptotically in upper complex half plane along some parabolas, and its displacement as big as the number n. By formula (38) for all fixed $\varepsilon > 0$ the branch of eigenvalues is situated along the parabola

$$\operatorname{Re}\lambda = \frac{(\operatorname{Im}\lambda)^2 l^2}{\varepsilon^2 + o(\varepsilon^2)} - \frac{\varepsilon^2 + o(\varepsilon^2)}{4l^2}, \tag{66}$$

rotated through the angle $\arg(-1+\varepsilon^2+2i\varepsilon)$.

6. Conclusion

In present work we consider some auxiliary spectral boundary value problems arising in theory of metamaterials. We prove that general problem has discrete spectrum situated in the sector $\arg\beta \leq \arg\lambda \leq \arg\alpha$. In the special one-dimensional case we find characteristic equation for eigenvalues λ_n and establish its localization in the narrow angles $\arg\alpha \leq \arg\lambda \leq \arg\alpha + \delta$ and $\arg\beta - \delta \leq \arg\lambda \leq \arg\beta$ $(\delta > 0)$ for $n \to \infty$. More precise, we find exact solutions of limit characteristic equations corresponding to each of these sectors and prove that asymptotic behavior of the eigenvalues coincides with these solutions which are parabolas with polar axis of symmetries $\arg\lambda = \arg\alpha$ and $\arg\lambda = \arg\beta$. For the interesting physical case $\alpha = 1$, $\sqrt{\beta} = i + \varepsilon$ we find asymptotic behavior of the eigenvalues λ_n for $\varepsilon \to +0$ depending on the number n.

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О спектральных свойствах некоторых вспомогательных краевых задач теории метаматериалов

Рассматривается новая спектральная краевая задача, возникающая в теории метаматериалов. В общем случае доказано, что ее спектр является дискретным и расположен в некотором секторе. В частном одномерном случае найдена локализация собственных значений, стремящихся к бесконечности, а также некоторые асимптотические формулы.

Ключевые слова: Спектральная краевая задача, дискретный спектр, локализация собственных значений, асимптотические формулы

Про спектральні властивості деяких допоміжних крайових задач теорії метаматеріалів

Розглядається нова спектральна крайова задача, що виникає в теорії метаматеріалів. У загальному випадку доведено, що її спектр є дискретним та розташованим у деякому секторі. У частному одномірному випадку знайдена локалізація власних значень, що збігаються до нескінченості, а також деякі асимптотичні формули.

Ключевые слова: Спектральная краевая задача, дискретный спектр, локализация собственных значений, асимптотические формулы