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COMPARISON OF ORLICZ, LORENTZ AND ORLICZ-LORENTZ SPACES

In this paper we investigate conditions imposed on Orlicz functions and Lorentz functions such that one Orlicz-Lorentz space is embedded to another or these spaces coincide. Similar results are showed for general rearrangement invariant spaces and, in particularly, for Orlicz and Lorentz spaces.

Keywords: rearrangement spaces, Orlicz-Lorentz spaces, comparison

Introduction

The theory of symmetric spaces dates back to the classical $\mathbf{L}_p$ spaces, $1 \leq p \leq \infty$. The theory was developed intensively during the last century; it contains many interesting and deep results that have important applications in various areas of the theory of functions and functional analysis. It applies in particularly, to the areas of interpolation of linear operators, ergodic theory, harmonic analysis and mathematical physics.

Last years much attention has been given to Orlicz-Lorentz spaces. From a general point of view It help us to investigate such rearrangement invariant spaces as Orlicz spaces, Lorentz spaces and $\mathbf{L}_{p,q}$ spaces. There are several ways of defining these spaces.

We can refer to [14], [15], [3], [4], [5], [6], [9] and to the references cited therein as well.

Our definition of Orlicz-Lorentz spaces is differ from another which is the basis in cited papers. In our opinion, it is more convenient to deal with issues related to the question embedding one Orlicz-Lorents space to another.

1. Preliminaries

Let μ be the Lebesgue measure on the positive semiaxis $[0, \infty)$, $\mathbf{L}_0$ the space of all μ-measurable almost everywhere finite functions f on $(0, \infty)$, $\mathbf{L}_p$, $1 \leq p \leq +\infty$ — Banach space of functions from $\mathbf{L}_0$, an integrable p degree.

A Banach space $\mathbf{E} \subset \mathbf{L}_0$ is called *rearrangement invariant* if

$$f \in \mathbf{L}_0\ ,\ g \in \mathbf{E},\ f^* \leq g^* \implies f \in \mathbf{E}\ ,\ \|f\|_{\mathbf{E}} \leq \|g\|_{\mathbf{E}}\ .$$

Here f^* denotes the decreasing right-continuous rearrangement of $|f|$. It can be defined as the right-continuous generalized inverse

$$f^*(x) := \inf\{y \in [0, +\infty) \colon \mathbf{n}_f(y) \leq x\}, \qquad x \in [0, \infty)$$

of the distribution function $\mathbf{n}_f$ of $|f|$, which is

$$\mathbf{n}_f(x) = \mu\{u \in (0, \infty) \colon |f(u)| > x\}\ ,$$

It is known (see [10], Ch. II, §4.1 or [12], Ch. 2.a), that for every rearrangement invariant space $\mathbf{E}$ there exist continuous inclusions

$$\mathbf{L}_1 \cap \mathbf{L}_\infty \subseteq \mathbf{E} \subseteq \mathbf{L}_1 + \mathbf{L}_\infty \subseteq \mathbf{L}_0.$$

Enclosed symmetric space. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ — are two symmetric spaces.

Theorem 1. *Let $\mathbf{E}_1 \subseteq \mathbf{E}_2$. Then the embedding*

$$i : \mathbf{E}_1 \ni f \to f \in \mathbf{E}_2$$

is continuous (bounded).

Proof. Let $\{f_n\}$ is the sequence in the space $\mathbf{E}_1$, such that

$$\|f_n - f\|_{\mathbf{E}_1} \to 0 \ \text{ and } \ \|f_n - g\|_{\mathbf{E}_2} \to 0.$$

Then we have $f_n \to f$ and $f_n \to g$ in measure and $f = i(f) = g$. Thus the graph is closed for embedding operator i. In accordance with the theorem on a closed operator, the operator i is bounded (continuous). □

Remark 1. The continuity of the embedding $\mathbf{E}_1 \subseteq \mathbf{E}_2$ means that

$$\|f\|_{\mathbf{E}_2} \leq c\|f\|_{\mathbf{E}_1}, \quad f \in \mathbf{E}_1.$$

for some $c > 0$.

As follows from the open mapping theorem, the space $\mathbf{E}_1$ is closed in the space $\mathbf{E}_2$ if and only if this embedding is open, i.e.

$$\|f\|_{\mathbf{E}_1} \leq c_1\|f\|_{\mathbf{E}_2}, f \in \mathbf{E}_1.$$

for some $c_1 > 0$.

2. Comparison of Orlicz spaces

Let $\Phi\colon [0,+\infty) \to [0,+\infty]$ be an Orlicz function, i.e., $\Phi(0)=0$, Φ is increasing left-continuous and convex. Assume also that Φ is nontrivial, i.e., $\Phi(x)>0$ and $\Phi(y)<\infty$ for some $x,y>0$. The derivative Φ' exists a.e., and it is assumed to be left-continuous with $\Phi'(x)=+\infty$ iff $\Phi(x)=+\infty$.

The *Orlicz space* $\mathbf{L}_\Phi$ is the set defined as follows

$$\mathbf{L}_\Phi := \left\{ f \in \mathbf{L}_0\colon \int\limits_0^\infty \Phi(f/a)\, d\mu < \infty \text{ for some } a>0 \right\},$$

equipped with the norm

$$\|f\|_{\mathbf{L}_\Phi} := \inf \left\{ a>0\colon \int\limits_0^\infty \Phi(|f|/a)\, d\mu \le 1 \right\}, \ f \in \mathbf{L}_0 ,$$

where $\inf \emptyset := \infty$.

Notice that this "slightly generalized" definition includes the spaces $\mathbf{L}_1$, $\mathbf{L}_\infty$ and also $\mathbf{L}_1 \cap \mathbf{L}_\infty$, $\mathbf{L}_1 + \mathbf{L}_\infty$ as the smallest and largest Orlicz spaces (see , [8], Ch. 2, [2], Ch. 2, §2.1 and also [16], [17], [18]).

The fundamental function of Orlicz space. Now we turn to the fundamental function $\varphi_{\mathbf{L}_\Phi}$ of the Orlicz space $\mathbf{L}_\Phi$.

Proposition 1. The fundamental function $\varphi_{\mathbf{L}_\Phi}$ of the Orlicz space $\mathbf{L}_\Phi$ is:

$$\varphi_{\mathbf{L}_\Phi}(x) = \left(\Phi^{-1}\left(x^{-1}\right)\right)^{-1}, \quad x>0. \tag{1}$$

Proof. For all $x>0$ and $a>0$ we have:

$$\mathcal{I}_\Phi\left(\frac{1}{a}\cdot 1_{[0,x]}\right) = \int\limits_0^\infty \Phi\left(\frac{1}{a}\cdot 1_{[0,x]}\right) dm = \int\limits_0^x \Phi\left(\frac{1}{a}\right) dm = x\,\Phi\left(\frac{1}{a}\right).$$

Therefore

$$\varphi_{\mathbf{L}_\Phi}(x) = \|1_{[0,x]}\|_{\mathbf{L}_\Phi} = \inf\left\{a>0 : x\,\Phi\left(\frac{1}{a}\right) \le 1\right\} =$$

$$= \inf\left\{a>0 : a \ge \left(\Phi^{-1}\left(\frac{1}{x}\right)\right)^{-1}\right\} = \left(\Phi^{-1}\left(\frac{1}{x}\right)\right)^{-1}.$$

□

Corollary 1. *The Orlicz function Φ is uniquely reconstructed from the fundamental function $\varphi_{\mathbf{L}_\Phi}$ of Orlicz space $\mathbf{L}_\Phi$, that is*

$$\Phi^{-1}(x) = \left(\varphi_{\mathbf{L}_\Phi}\left(\frac{1}{x}\right)\right)^{-1}, \ x>0,$$

and $\Phi = (\Phi^{-1})^{-1}$ *is the inverse function of the function* Φ^{-1}.

The embedding $\mathbf{L}_{\Phi_1} \subseteq \mathbf{L}_{\Phi_2}$ of Orlicz space $\mathbf{L}_{\Phi_1}$ in the Orlicz space $\mathbf{L}_{\Phi_2}$ can be described in terms of corresponding Orlicz functions Φ_1 and Φ_2.

Recall, that the embedding space $\mathbf{L}_{\Phi_1}$ in the space $\mathbf{L}_{\Phi_2}$ is always bounded, i.e.

$$\|f\|_{\mathbf{L}_{\Phi_2}} \le c\|f\|_{\mathbf{L}_{\Phi_1}}, \quad f \in \mathbf{L}_{\Phi_1},$$

for some $c > 0$ (Theorem 1).

Definition 1. Let Φ_1 и Φ_2 are Orlicz functions. It say that

1). Φ_1 is *majorizes* the Φ_2 in 0 ($\Phi_1 \succ_0 \Phi_2$), if there are exist positive numbers a, b, x_0 such that the inequality

$$\Phi_2(x) \le b\ \Phi_1(ax)$$

is satisfied for all $0 \le x \le x_0$.

2). Φ_1 is *majorizes* Φ_2 on ∞ ($\Phi_1 \succ_\infty \Phi_2$), if there are exist positive numbers a, b, x_0 such that the inequality

$$\Phi_2(x) \le b\ \Phi_1(ax)$$

is satisfied for all $x \ge x_0$.

3). Φ_1 is *majorizes* Φ_2 ($\Phi_1 \succ \Phi_2$), if $\Phi_1 \succ_0 \Phi_2$ and $\Phi_1 \succ_\infty \Phi_2$.

Remark 2. 1). We can take $b = 1$ in the conditions 1) and 2).

2). The condition $\Phi_1 \succ \Phi_2$ has the form

$$\Phi_2(x) \le b\,\Phi_1(ax),\ x \ge 0$$

for some $b > 0$ and $a > 0$.

Theorem 2. *Let* Φ_1 *and* Φ_2 *are Orlicz functions, the functions* $\varphi_{\mathbf{L}_{\Phi_1}}$ *and* $\varphi_{\mathbf{L}_{\Phi_2}}$ *are fundamental function of corresponding Orlicz spaces* $\mathbf{L}_{\Phi_1}$ *и* $\mathbf{L}_{\Phi_2}$. *Then the following conditions are equivalent:*

1). $\Phi_1 \succ \Phi_2$*;*

2). $\mathbf{L}_{\Phi_1} \subseteq \mathbf{L}_{\Phi_2}$*;*

3). $\|\cdot\|_{\mathbf{L}_{\Phi_2}} \le a\|\cdot\|_{\mathbf{L}_{\Phi_1}}$ *for some* $a > 0$*;*

4). $\varphi_{\mathbf{L}_{\Phi_2}} \le a\varphi_{\mathbf{L}_{\Phi_1}}$ *for some* $a > 0$*;*

5). $\Phi_2(x) \le \Phi_1(ax)$ *for some* $a > 0$ *and for all* $x > 0$.

Proof. 1) $\Longrightarrow$ 2). The condition $\Phi_1 \succ \Phi_2$ has the form:

$$\Phi_2(x) \leq b\, \Phi_1(ax),\ \ x \geq 0$$

for some $b > 0$ and $a > 0$.

If $f \in \mathbf{L}_{\Phi_1}$, then for some $c > 0$ we have

$$\mathcal{I}_{\Phi_1}\left(\frac{f}{c}\right) = \int\limits_0^\infty \Phi_1\left(\frac{|f|}{c}\right) dm < \infty.$$

Therefore

$$\mathcal{I}_{\Phi_2}\left(\frac{f}{ac}\right) = \int\limits_0^\infty \Phi_2\left(\frac{|f|}{ac}\right) dm \leq b \int\limits_0^\infty \Phi_1\left(\frac{|f|}{c}\right) dm = b\, \mathcal{I}_{\Phi_1}\left(\frac{f}{c}\right) < \infty,$$

i.e. $f \in \mathbf{L}_{\Phi_2}$, whence $\mathbf{L}_{\Phi_1} \subseteq \mathbf{L}_{\Phi_2}$.

2) $\Longrightarrow$ 3). It is following from Proposition 1.

3) $\Longrightarrow$ 4). We use the function $f = 1_{[0,x]}$ in the condition 3). Thus we have 4).

4) $\Longrightarrow$ 5). We use the formula (1) for the fundamental function of the Orlicz space. Thus

$$\left(\Phi_2^{-1}\left(x^{-1}\right)\right)^{-1} \leq a \left(\Phi_1^{-1}\left(x^{-1}\right)\right)^{-1},\ \ \ x > 0.$$

Suppose $x^{-1} = \Phi_2(y)$, then

$$\Phi_1^{-1}\left(\Phi_2(y)\right) \leq a\, \Phi_2^{-1}\left(\Phi_2(y)\right) = ay,\ \ y > 0$$

or

$$\Phi_2(y) \leq \Phi_1(ay),\ \ y > 0.$$

5) $\Longrightarrow$ 1). It is obvious. □

Definition 2. Two Orlicz functions Φ_1 and Φ_2 are called *equivalent* ($\Phi_1 \approx \Phi_2$), if the conditions $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$ are hold.

Corollary 2. *Let Φ_1 and Φ_2 are Orlicz functions. Then the following conditions are equivalent:*

1). $\Phi_1 \approx \Phi_2$*;*
2). $\mathbf{L}_{\Phi_1} = \mathbf{L}_{\Phi_2}$ *(as sets);*
3). *Norms* $\|\cdot\|_{\mathbf{L}_{\Phi_1}}$ *and* $\|\cdot\|_{\mathbf{L}_{\Phi_2}}$ *are equivalent, i.e.*

$$a_1\|f\|_{\mathbf{L}_{\Phi_1}} \leq \|f\|_{\mathbf{L}_{\Phi_2}} \leq a_2\|f\|_{\mathbf{L}_{\Phi_1}}$$

for all f and some $a_1 > 0$, $a_2 > 0$;

4). *Fundamental functions* $\varphi_{\mathbf{L}_{\Phi_1}}$ *and* $\varphi_{\mathbf{L}_{\Phi_2}}$ *are equivalent, i.e.*

$$a_1\varphi_{\mathbf{L}_{\Phi_1}}(x) \le \varphi_{\mathbf{L}_{\Phi_2}}(x) \le a_2\varphi_{\mathbf{L}_{\Phi_1}}(x)$$

for all f *and some* $a_1 > 0$, $a_2 > 0$;

5). *Functions* Φ_1 *and* Φ_2 *are equivalent in the next sense:*

$$\Phi_1(a_1 x) \le \Phi_2(x) \le \Phi_1(a_2 x), \ x \ge 0$$

for some $a_1 > 0$ *and* $a_2 > 0$.

Remark 3. Constants a_1 and a_2 in all conditions 3), 4) and 5) of previous Theorem 2 are the same. For example, the condition 5) is equivalent of the condition

$$\frac{1}{a_2}\Phi_1^{-1}(y) \le \Phi_2^{-1}(y) \le \frac{1}{a_1}\Phi_1^{-1}(y), \ y > 0$$

(by $y = \Phi_2(x)$). So

$$a_1\left(\Phi_1^{-1}\left(\frac{1}{x}\right)\right)^{-1} \le \left(\Phi_2^{-1}\left(\frac{1}{x}\right)\right)^{-1} \le a_2\left(\Phi_1^{-1}\left(\frac{1}{x}\right)\right)^{-1}, \quad x > 0,$$

i.e.

$$a_1\varphi_{\mathbf{L}_{\Phi_1}}(x) \le \varphi_{\mathbf{L}_{\Phi_2}}(x) \le a_2\varphi_{\mathbf{L}_{\Phi_1}}(x), \quad x > 0.$$

3. Comparison of Lorentz spaces

Let W be an increasing function on $[0, +\infty)$ such that: $W(0) = 0$, W is concave on $(0, +\infty)$, and $W(x) > 0$ for some $x > 0$. Then W is absolutely continuous on the open interval $(0, \infty)$ with the decreasing density function $W'(x)$, $x > 0$, while $W(0+)$ may be positive.

The Lorentz space $\mathbf{\Lambda}_W$ is defined as

$$\mathbf{\Lambda}_W := \{f \in \mathbf{L}_0 \colon \|f\|_{\mathbf{\Lambda}_W} < +\infty\}$$

with the norm

$$\|f\|_{\mathbf{\Lambda}_W} := \int_0^\infty f^*(x)\, dW(x) = f^*(0)W(0+) + \int_0^\infty f^*(x)\, W'(x)\, dx < \infty\,,$$

where $+\infty \cdot 0 = 0$ (see [10], Ch. II, §5.1, and also [12], Ch. 2, [13] and references therein.

The Stieltjes integral $\int_0^\infty f^*(x)\, dW(x)$ has an atomic part $f^*(0)\, W(0+)$ in the case $W(0+) > 0$. The Lorentz spaces are maximal rearrangement invariant spaces with respect to the norm $\|\cdot\|_{\mathbf{\Lambda}_W}$.

The norm $\|f\|_{\mathbf{\Lambda}_W}$ of function $f \in \mathbf{\Lambda}_W$ can be written as

$$\|f\|_{\mathbf{\Lambda}_W} = \int\limits_0^\infty W(\eta_{|f|}(x))dx = \int\limits_0^\infty W \circ \eta_{f^*} dm. \tag{2}$$

Indeed, using the substitution $x = \eta_{f^*}(y),\ \ y = f^*(x)$, we get:

$$\|f\|_{\mathbf{\Lambda}_W} = \int\limits_0^\infty f^* dW = f^*(x)W(x)|_0^\infty - \int\limits_0^\infty W(x)df^*(x) =$$

$$= \int\limits_\infty^0 W(x)df^*(x) = \int\limits_0^\infty W(\eta_{f^*}(y))dy = \int\limits_0^\infty W \circ \eta_{f^*} dm.$$

By this definition $\mathbf{\Lambda}_W \subseteq \mathbf{L}_\infty$ if $W(0+) > 0$, and $\mathbf{\Lambda}_W \supseteq \mathbf{L}_\infty$ if $W(+\infty) := \lim\limits_{x\to\infty} W(x) < +\infty$. Whence $\mathbf{\Lambda}_W = \mathbf{L}_\infty$ if both the conditions $W(0+) > 0$ and $W(+\infty) < +\infty$ hold.

We calculate the fundamental function of Lorentz space $\mathbf{\Lambda}_W$:

$$\varphi_{\mathbf{\Lambda}_W}(x) = \int\limits_0^\infty (1_{[0,x]})^* dW = \int\limits_0^x dW = W(x)$$

for all $x > 0$ and $\varphi_{\mathbf{\Lambda}_W}(0) = W(0) = 0$, i.e.,

$$\varphi_{\mathbf{\Lambda}_W} = W. \tag{3}$$

Theorem 3. *Let W_1 u W_2 are two Lorentz function. Then the following conditions are equivalent:*

1). $\mathbf{\Lambda}_{W_1} \subseteq \mathbf{\Lambda}_{W_2}$*;*
2). $W_2(x) \le c\,W_1(x)$ *for all* $x \ge 0$ *and some* $c > 0$.

Proof. 1) $\implies$ 2). Let $\mathbf{\Lambda}_{W_1} \subseteq \mathbf{\Lambda}_{W_2}$. Then, there exists $c > 0$ such that

$$\|f\|_{\mathbf{\Lambda}_{W_2}} \le c\,\|f\|_{\mathbf{\Lambda}_{W_1}},$$

(from Proposition 1).

Therefore

$$W_2(x) = \varphi_{\mathbf{\Lambda}_{W_2}}(x) = \|1_{[0,x]}\|_{\mathbf{\Lambda}_{W_2}} \le c\|1_{[0,x]}\|_{\mathbf{\Lambda}_{W_1}} = c\varphi_{\mathbf{\Lambda}_{W_1}}(x) = c\,W_1(x).$$

2) $\implies$ 1). Backwards, let $W_2(x) \le c\,W_1(x)$ for all $x \ge 0$ and some $c > 0$ and $f \in \mathbf{\Lambda}_{W_1}$. Then

$$\|f\|_{\mathbf{\Lambda}_{W_1}} = \int\limits_0^\infty W_1(\eta_{|f|}(x))dx < \infty,$$

(from formula (2)).

Therefore

$$\|f\|_{\mathbf{\Lambda}_{W_2}} = \int_0^\infty W_2(\eta_{|f|}(x))dx \le c\int_0^\infty W_1(\eta_{|f|}(x))dx < \infty,$$

and $f \in \mathbf{\Lambda}_{W_2}$. Thus $\mathbf{\Lambda}_{W_1} \subseteq \mathbf{\Lambda}_{W_2}$.

□

Corollary 3. *Let W_1 and W_2 are two Lorentz functions. Then, the following conditions are equivalent*

1). $\mathbf{\Lambda}_{W_1} = \mathbf{\Lambda}_{W_2}$*;*

2). $c_1 W_1(x) \le W_2(x) \le c_2\, W_1(x)$ *for all* $x \ge 0$ *and some* $c_1, c_2 > 0$.

4. Comparison of Orlicz-Lorentz spaces

The rearrangement invariant spaces $\mathbf{\Lambda}_{\Phi,W}$, can be defined by

$$\mathbf{\Lambda}_{\Phi,W} := \{f \in \mathbf{L}_0 \colon \mathcal{I}_{\Phi,W}(f/a) < \infty \text{ for some } a > 0\}\ ,$$

with the norm

$$\|f\|_{\mathbf{\Lambda}_{\Phi,W}} := \inf\{a > 0 \colon \mathcal{I}_{\Phi,W}(f/a) \le 1\}\ ,$$

where

$$\mathcal{I}_{\Phi,W}(f) := \int_0^\infty \Phi(f^*(x))\, dW(x)\ ,\ f \in \mathbf{L}_0.$$

The functions Φ and W is Orlicz and Lorentz functions consequently.

We can refer to [14], [15], [3], [4], [5], [6], [9] and to the references cited therein as well.

Remark 4. As $f \in \mathbf{\Lambda}_{\Phi,W}$ if and only if when exist such $a > 0$ that

$$\mathcal{I}_{\Phi,W}\left(\frac{f^*}{a}\right) = \int_0^\infty \Phi\left(\frac{f^*(x)}{a}\right) dW(x) = \left\|\Phi\left(\frac{f^*(x)}{a}\right)\right\|_{\mathbf{\Lambda}_W} < \infty,$$

so

$$f \in \mathbf{\Lambda}_{\Phi,W} \Longleftrightarrow \Phi\left(\frac{f^*}{a}\right) \in \mathbf{\Lambda}_{\Phi,W} \quad \text{для некоторого} \quad a > 0.$$

We obtain the fundamental function of Orlicz-Lorentz space $\mathbf{\Lambda}_{\Phi,W}$.

Proposition 2. The fundamental function $\varphi_{\mathbf{\Lambda}_{\Phi,W}}$ of Orlicz-Lorentz space $\mathbf{\Lambda}_{\Phi,W}$ is:

$$\varphi_{\mathbf{\Lambda}_{\Phi,W}}(x) = \left(\Phi^{-1}\left(\frac{1}{W(x)}\right)\right)^{-1}, \quad x > 0. \tag{4}$$

Proof. For any $x > 0$ and $a > 0$ we have:

$$\mathcal{I}_{\Phi,W}\left(\frac{1}{a}\cdot 1_{[0,x]}\right) = \int\limits_0^\infty \Phi\left(\frac{1}{a}\cdot 1_{[0,x]}(t)\right) dW(t) = \int\limits_0^x \Phi\left(\frac{1}{a}\right) dW(t) = W(x)\,\Phi\left(\frac{1}{a}\right).$$

So

$$\varphi_{\mathbf{\Lambda}_{\Phi,W}}(x) = \|1_{[0,x]}\|_{\mathbf{\Lambda}_{\Phi,W}} = \inf\left\{a > 0 : W(x)\,\Phi\left(\frac{1}{a}\right) \le 1\right\} =$$

$$= \inf\left\{a > 0 : a \ge \left(\Phi^{-1}\left(\frac{1}{W(x)}\right)\right)^{-1}\right\} = \left(\Phi^{-1}\left(\frac{1}{W(x)}\right)\right)^{-1}.$$

□

Theorem 4. *Let Φ_1 and Φ_2 are Orlicz functions, W is Lorentz functions, the functions $\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}$ and $\varphi_{\mathbf{\Lambda}_{\Phi_2,W}}$ are fundamental functions of corresponding Orlicz-Lorentz spaces $\mathbf{\Lambda}_{\Phi_1,W}$ and $\mathbf{\Lambda}_{\Phi_2,W}$. Then the following conditions are equivalent:*

1). $\Phi_1 \succ \Phi_2$;
2). $\mathbf{\Lambda}_{\Phi_1,W} \subseteq \mathbf{\Lambda}_{\Phi_2,W}$;
3). $\|\cdot\|_{\mathbf{\Lambda}_{\Phi_2,W}} \le c\|\cdot\|_{\mathbf{\Lambda}_{\Phi_1,W}}$ *for some* $c > 0$;
4). $\varphi_{\mathbf{\Lambda}_{\Phi_2,W}} \le c\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}$ *for some* $c > 0$;
5). $\Phi_2(x) \le \Phi_1(cx)$ *for some* $c > 0$ *and for all* $x > 0$.

Proof. 1) $\Longrightarrow$ 2). The condition $\Phi_1 \succ \Phi_2$ has the form:

$$\Phi_2(x) \le b\ \Phi_1(ax),\ x \ge 0$$

for some $b > 0$ and $a > 0$.

If $f \in \mathbf{\Lambda}_{\Phi_1,W}$, then for some $c > 0$ we have

$$\mathcal{I}_{\Phi_1,W}\left(\frac{f}{c}\right) = \int\limits_0^\infty \Phi_1\left(\frac{f^*}{c}\right) dW < \infty.$$

Therefore

$$\mathcal{I}_{\Phi_2,W}\left(\frac{f}{ac}\right) = \int\limits_0^\infty \Phi_2\left(\frac{f^*}{ac}\right) dW \le b\int\limits_0^\infty \Phi_1\left(\frac{f^*}{c}\right) dW = b\,\mathcal{I}_{\Phi_1,W}\left(\frac{f}{c}\right) < \infty,$$

i.e. $f \in \mathbf{\Lambda}_{\Phi_2,W}$, whence $\mathbf{\Lambda}_{\Phi_1,W} \subseteq \mathbf{\Lambda}_{\Phi_2,W}$.

2) $\Longrightarrow$ 3). It is follows from 1.

3) $\Longrightarrow$ 4). Let $\|f\|_{\mathbf{\Lambda}_{\Phi_2,W}} \le c\|f\|_{\mathbf{\Lambda}_{\Phi_1,W}}$ for some $c > 0$ and for any $f \in \mathbf{\Lambda}_{\Phi_1,W}$. Using the function $f = 1_{[0,x]}$, we get:

$$\varphi_{\mathbf{\Lambda}_{\Phi_2,W}}(x) = \|1_{[0,x]}\|_{\mathbf{\Lambda}_{\Phi_2,W}} \le c\|1_{[0,x]}\|_{\mathbf{\Lambda}_{\Phi_1,W}} = c\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}(x).$$

4) $\Longrightarrow$ 5). We use the formula (4) for the fundamental function of Orlicz-Lorentz space $\mathbf{\Lambda}_{\Phi,W}$, we have:

$$\left(\Phi_2^{-1}\left(\frac{1}{W(x)}\right)\right)^{-1} \leq c\left(\Phi_1^{-1}\left(\frac{1}{W(x)}\right)\right)^{-1}, \quad x > 0.$$

Suppose $\frac{1}{W(x)} = t$, then

$$\left(\Phi_2^{-1}(t)\right)^{-1} \leq c\left(\Phi_1^{-1}(t)\right)^{-1}, \quad t > 0,$$

therefore

$$\Phi_1^{-1}(t) \leq c\Phi_2^{-1}(t).$$

Suppose $t = \Phi_2(y)$. Then

$$\Phi_1^{-1}\left(\Phi_2(y)\right) \leq c\,\Phi_2^{-1}\left(\Phi_2(y)\right) = cy, \; y > 0$$

or

$$\Phi_2(y) \leq \Phi_1(cy), \; y > 0.$$

5) $\Longrightarrow$ 1). It is obvious. □

Corollary 4. *Let Φ_1 and Φ_2 are Orlicz functions, W is Lorentz function. the functions $\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}$ and $\varphi_{\mathbf{\Lambda}_{\Phi_2,W}}$ are fundamental functions of corresponding Orlicz-Lorentz spaces $\mathbf{\Lambda}_{\Phi_1,W}$ and $\mathbf{\Lambda}_{\Phi_2,W}$. Then the following conditions are equivalent:*

1). $\Phi_1 \approx \Phi_2$;

2). $\mathbf{\Lambda}_{\Phi_1,W} = \mathbf{\Lambda}_{\Phi_2,W}$ *(as sets);*

3). *Norms $\|\cdot\|_{\mathbf{\Lambda}_{\Phi_1,W}}$ u $\|\cdot\|_{\mathbf{\Lambda}_{\Phi_2,W}}$ are equivalent, i.e.*

$$a_1\|f\|_{\mathbf{\Lambda}_{\Phi_1,W}} \leq \|f\|_{\mathbf{\Lambda}_{\Phi_2,W}} \leq a_2\|f\|_{\mathbf{\Lambda}_{\Phi_1,W}}$$

for all f and some $a_1 > 0$, $a_2 > 0$;

4). *Fundamental functions $\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}$ and $\varphi_{\mathbf{\Lambda}_{\Phi_2,W}}$ are equivalent, i.e.*

$$a_1\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}(x) \leq \varphi_{\mathbf{\Lambda}_{\Phi_2,W}}(x) \leq a_2\varphi_{\mathbf{\Lambda}_{\Phi_1,W}}(x)$$

for all f and some $a_1 > 0$, $a_2 > 0$;

5). *Functions Φ_1 and Φ_2 are equivalent in the next sense:*

$$\Phi_1(a_1x) \leq \Phi_2(x) \leq \Phi_1(a_2x), \; x \geq 0$$

for some $a_1 > 0$ and $a_2 > 0$.

Theorem 5. *Let W_1 and W_2 are Lorentz functions and Φ is Orlicz function.*

1). *If $W_2(x) \leq c\,W_1(x)$ for all $x \geq 0$ and some $c > 0$, then $\mathbf{\Lambda}_{\Phi,W_1} \subseteq \mathbf{\Lambda}_{\Phi,W_2}$;*

2). *If the space $\mathbf{\Lambda}_{\Phi,W_1}$ is normally embedded in the space $\mathbf{\Lambda}_{\Phi,W_2}$ with embedded constant $c \leq 1$, mo $W_2(x) \leq c\,W_1(x)$ for all $x \geq 0$.*

Proof. 1) $\implies$ 2). As the space $\mathbf{\Lambda}_{\Phi,W_1}$ is normally embedded in the space $\mathbf{\Lambda}_{\Phi,W_2}$, so

$$\|f\|_{\mathbf{\Lambda}_{\Phi,W_2}} \le c\,\|f\|_{\mathbf{\Lambda}_{\Phi,W_1}},$$

where $c \le 1$. Therefore

$$\varphi_{\mathbf{\Lambda}_{\Phi,W_2}}(x) = \|1_{[0,x]}\|_{\mathbf{\Lambda}_{\Phi,W_2}} \le c\|1_{[0,x]}\|_{\mathbf{\Lambda}_{\Phi,W_1}} = c\varphi_{\mathbf{\Lambda}_{\Phi,W_1}}(x),$$

i.e.

$$\left(\Phi^{-1}\left(\frac{1}{W_2(x)}\right)\right)^{-1} \le c\left(\Phi^{-1}\left(\frac{1}{W_1(x)}\right)\right)^{-1}.$$

Consequently,

$$\Phi^{-1}\left(\frac{1}{W_1(x)}\right) \le c\Phi^{-1}\left(\frac{1}{W_2(x)}\right)$$

The function Φ is increasing convex on $(0,\infty)$. So

$$\Phi\left(\Phi^{-1}\left(\frac{1}{W_1(x)}\right)\right) \le \Phi\left(c\Phi^{-1}\left(\frac{1}{W_2(x)}\right)\right) \le c\Phi\left(\Phi^{-1}\left(\frac{1}{W_2(x)}\right)\right),$$

i.e.,

$$\frac{1}{W_1(x)} \le c\frac{1}{W_2(x)},$$

Whence

$$W_2(x) \le c\,W_1(x)$$

for all $x \ge 0$.

2) $\implies$ 1). Conversely, let $W_2(x) \le c\,W_1(x)$ for all $x \ge 0$ and some $c > 0$ and let $f \in \mathbf{\Lambda}_{\Phi,W_1}$. So $\Phi\left(\frac{f^*}{a}\right) \in \mathbf{\Lambda}_{W_1}$ for some $a > 0$ and, using the formula (2),

$$\left\|\Phi\left(\frac{f^*}{a}\right)\right\|_{\mathbf{\Lambda}_{W_1}} = \int\limits_0^\infty W_1\left(\eta_{\Phi\left(\frac{f^*}{a}\right)}(x)\right)dx < \infty.$$

Therefore

$$\left\|\Phi\left(\frac{f^*}{a}\right)\right\|_{\mathbf{\Lambda}_{W_2}} = \int\limits_0^\infty W_2\left(\eta_{\Phi\left(\frac{f^*}{a}\right)}(x)\right)dx \le c\int\limits_0^\infty W_1\left(\eta_{\Phi\left(\frac{f^*}{a}\right)}(x)\right)dx < \infty,$$

whence $f \in \mathbf{\Lambda}_{\Phi,W_2}$. Thus, $\mathbf{\Lambda}_{\Phi,W_1} \subseteq \mathbf{\Lambda}_{\Phi,W_2}$.

□

Corollary 5. *Let Φ_1 u Φ_2 are Orlicz functions, and W_1 u W_2 are Lorentz functions. If*

1). $\Phi_1 \approx \Phi_2$;

2). $c_1 W_1(x) \le W_2(x) \le c_2\,W_1(x)$ *for all* $x \ge 0$ *and some* $c_1, c_2 > 0$,

so $\mathbf{\Lambda}_{\Phi_1,W_1} = \mathbf{\Lambda}_{\Phi_2,W_2}$;

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Сравнение пространств Орлича, Лоренца и Орлича-Лоренца

В работе рассматриваются условия на функции Орлича и функции Лоренца, при выполнении которых одно пространство Орлича-Лоренца содержится в другом или когда эти пространства совпадают. Приведены аналогичные результаты для общих перестановочно инвариантных пространств и, в частности, для пространств Орлича и Лоренца.

Ключевые слова: перестановочно инвариантные пространства, пространства Орлича-Лоренца, сравнение пространств.

Порівняння просторів Орліча, Лоренца та Орліча-Лоренца

В роботі розглядаються умови на функції Орліча та на функції Лоренца, при виконанні яких один простір Орліча-Лоренца є у іншому або коли ці простори співпадають. Наведено аналогічні результати для загальних перестановочно інваріантних просторів та, зокрема, для просторів Орліча та Лореца.

Ключові слова: перестановочно інваріантні простори, простори Орліча-Лоренца, порівняння просторів.