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SELFADJOINTNESS OF CAUCHY SINGULAR INTEGRAL OPERATOR

Let G_+ be a finite-connected domain bounded by the rectifiable curve $C = \partial G_+$, $G_- = \mathbb{C} \setminus \operatorname{clos} G_+$ and $\infty \in G_-$. Suppose also that $w(z)$, $z \in C$ is a nonnegative weight such that $w(z) \not\equiv 0$ on each connected component of the curve C. For any $f \in L^2(C, |dz|)$, we denote by $f_\pm(z)$, $z \in C$ the angular boundary values of the Cauchy transform

$$\mathcal{K}(f, \lambda) := \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{z - \lambda} dz, \quad \lambda \notin C$$

from the domains $G_\pm$, respectively. The well-known David's theorem [1] says that the mappings $P_\pm \colon f \mapsto \pm f_\pm$ are bounded linear operators in $L^2(C, |dz|)$ if and only if the curve C is a Carleson curve. Moreover, the operators $P_\pm$ are bounded in $L^2(C, w(z)|dz|)$ if and only if the weight w is a Mackenhoupt weight (see, e.g. [2]), where the vector space $L^2(C, w(z)|dz|)$ is endowed with the inner product

$$(f, g)_{L^2(C,w)} = \frac{1}{2\pi} \int\limits_C f(z)\, \overline{g(z)}\, w(z)\, |dz|,$$

and $|dz|$ is the arc-length measure. In the sequel, we always assume that C is a Carleson curve.

In the paper we are interested in finding necessary and sufficient conditions for selfadjointness of the projections $P_\pm$ and therefore (note that $P_+ + P_- = I$) for selfadjointness of the corresponding Cauchy singular integral operator

$$\mathcal{K}_S(f, \lambda) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{C(\lambda,\varepsilon)} \frac{f(z)}{z - \lambda} dz = \frac{1}{2} \left((P_+ f)(\lambda) - (P_- f)(\lambda) \right),$$

where $\lambda \in C$ and $C(\lambda, \varepsilon) = \{z \in C \colon |z - \lambda| > \varepsilon\}$. For simple connected domains (i.e., when C is a simple closed curve) N.Krupnik [3] has established the criterion: *the bounded operator $\mathcal{K}_S$ is selfadjoint if and only if C is a circle and $w(z) \equiv const$* (see also [4] for previous results and [5] for applications). Our extension of Krupnik's criterion to the case of multiply-connected domains is a consequence of the following theorem, which is a slightly more strong assertion than Krupnik's result.

Theorem. *Let* $w \in L^1(C, |dz|)$. *The following conditions are equivalent:*
1) $\forall \lambda \in G_+ \, \forall \mu \in G_- \; (\frac{1}{z-\lambda}, \frac{1}{z-\mu})_{L^2(C,w)} = 0$;
2) the curve C *is a circle and* $w(z) \equiv const$.

Proof. The implication 2) $\Longrightarrow$ 1) is obvious.
1) $\Longrightarrow$ 2). Let $w(z)\,|dz| = k(z)\,dz$. Obviously, $k \in L^1(C, |dz|)$. Then $\forall \lambda \in G_+$

$$0 = -\lim_{\mu\to\infty} \int\limits_C \frac{1}{z-\lambda} \cdot \frac{\overline{\mu}}{\overline{z}-\overline{\mu}} k(z)\,dz = \int\limits_C \frac{k(z)}{z-\lambda} dz .$$

By Smirnov's theorem [6], we get $k(z) \in E^1(G_-)$. For the same reason, we have $\forall \lambda \in G_+ \, \forall \mu \in G_- \, \forall n \in \mathbb{N}$

$$0 = \int\limits_C \frac{1}{z-\lambda} \cdot \frac{k(z)}{(\overline{z}-\overline{\mu})^n} dz \quad \Longrightarrow \quad \frac{k(z)}{(\overline{z}-\overline{\mu})^n} \in E^1(G_-) .$$

Let f be defined by $f(z) = \overline{z}$, $z \in C$. Evidently, we have

$$f(z) = \overline{\mu} + \frac{k(z)}{g(z)}, \quad z \in C, \quad \text{where} \quad g(z) = \frac{k(z)}{\overline{z}-\overline{\mu}} \in E^1(G_-)$$

and therefore the function $f(z)$ admits meromorphic continuation into the domain G_-. Since $\forall n \in \mathbb{N} \; \frac{k(z)}{(f(z)-\overline{\mu})^n} \in E^1(G_-)$, we get $f : G_- \to \overline{\operatorname{clos} G_+}$ and $f(z) \in H^\infty(G_-)$.

The curve C can be represented in the form $C = \cup_{k=0}^n C_k$, where C_k are simple closed contours. We have $G_+ = \cap_{k=0}^n G_{k+}$, $G_- = \cup_{k=0}^n G_{k-}$, and $\infty \in G_{0-}$, where $G_{0+} = \operatorname{Int} C_0$, $G_{0-} = \operatorname{Ext} C_0$, $G_{k+} = \operatorname{Ext} C_k$, $G_{k-} = \operatorname{Int} C_k$, $k = \overline{1,n}$. Evidently, $f : G_{0-} \to \overline{\operatorname{clos} G_{0+}}$ and $f(z) \in H^\infty(G_{0-})$. Besides, $f(z) = \overline{z}$ is an one-to-one correspondence between C_0 and $\overline{C_0} = \{\bar{z}\colon z \in C_0\}$. Hence the function f is a conformal mapping of G_0 onto $\overline{G_{0+}}$.

Let $z = \varphi(\zeta)$ be a conformal mapping of the unit disk $\mathbb{D}$ onto G_{0+} and $\zeta = \psi(z)$ be its inverse. Consider the function $\psi_\infty(z) = \overline{\psi(\overline{f(z)})}$ and its inverse $z = \varphi_\infty(\zeta)$. It is clear that φ_∞ is a conformal mapping of $\mathbb{D}$ onto G_{0-}. Further, we have

$$\psi_\infty(z) = \overline{\psi(\overline{\overline{z}})} = \overline{\psi(z)}, \; z \in C_0; \quad \psi_\infty(\varphi(\zeta)) = \overline{\psi(\varphi(\zeta))} = \overline{\zeta}, \; |\zeta| = 1$$

and $\varphi(\zeta) = \varphi_\infty(\overline{\zeta})$. Therefore the function

$$\Phi(\zeta) = \begin{cases} \varphi(\zeta), & |\zeta| \le 1, \\ \varphi_\infty(1/\zeta), & |\zeta| \ge 1 \end{cases}$$

is a conformal mapping of the whole complex plane $\mathbb{C}$ onto itself and we obtain that $\Phi(\zeta) = \frac{a\zeta+b}{c\zeta+d}$, and C_0 is a circle.

The curves C_k, $k=\overline{1,n}$ are circles too. This claim can be reduced to the case of C_0: it follows easily from the observation that the operator

$$C_\varphi\colon L^2(\varphi(C), w(\tilde z)|d\tilde z|) \to L^2(C, w(\varphi(z))|dz|)\,,$$
$$(C_\varphi f(\cdot))(z) := \sqrt{\varphi'(z)}f(\varphi(z))\,, \quad z\in C\,, \quad f\in L^2(\varphi(C), w(\tilde z)|d\tilde z|)$$

is an unitary operator and from the straightforward computation

$$(C_\varphi f(\cdot))(z) = \frac{c\lambda+d}{\sqrt{ad-bc}}\cdot\frac{1}{z-\lambda}\,, \quad \tilde z = \varphi(z) = \frac{az+b}{cz+d}\,, \quad f(\tilde z) = \frac{1}{\tilde z - \varphi(\lambda)}\,.$$

Without loss of generality we can assume that C_0 is the unit circle. Other curves C_k are also circles (with centers a_k and radii r_k). By the same argument as above, $\forall\,\mu\in G_-\ \forall\,\lambda\in G_+$

$$0 = \int\limits_C \frac{1}{z-\mu}\cdot\frac{k(z)}{\overline{z}-\overline{\lambda}}\,dz \quad\Longrightarrow\quad h(z) = \frac{k(z)}{\overline{z}-\overline{\lambda}} \in E^1(G_+)\,.$$

Since $k(z)\in E^1(G_-)$, the function h admits the meromorphic continuation $h(z) = \dfrac{k(z)}{\frac{1}{z}-\overline{\lambda}}$ into the domain $G_{0-}=\{z:|z|>1\}$ and $h(z) = \dfrac{k(z)}{\overline{a}_k + \frac{r_k^2}{z-a_k} - \overline{\lambda}}$ into the domains $G_{k-}=\{z:|z-a_k|<r_k\}$. This meromorphic continuation has only simple poles at the points $b_0=\frac{1}{\overline{\lambda}}\in G_{0-}$ and $b_k = a_k + \frac{r_k^2}{\overline{\lambda}-\overline{a}_k}\in G_{k-}$. Therefore, we have

$$h(z) - \frac{c_0}{z-b_0} - \frac{c_1}{z-b_1}\cdots - \frac{c_n}{z-b_n} \in E^1(G_+)\cap E^1(G_-) = \{0\}\,,$$

where $c_k = c_k(\lambda) = \operatorname{res}(h(z),b_k)$, $k=\overline{0,n}$. Hence,

$$k(z) = c_0\frac{\overline{z}-\overline{\lambda}}{z-b_0} + c_1\frac{\overline{z}-\overline{\lambda}}{z-b_1} + \ldots + c_n\frac{\overline{z}-\overline{\lambda}}{z-b_n}\,, \quad z\in C\,.$$

In particular,

$$k(z) = -\frac{\overline{\lambda}}{z}\left(c_0 + c_1\frac{z-b_0}{z-b_1} + \ldots + c_n\frac{z-b_0}{z-b_n}\right)\,, \quad z\in C_0\,.$$

Since coefficients c_k depend analytically on $\overline{\lambda}$ and the functions $b_k=b_k(\lambda)$ are not constants, we obtain $c_k=0$, $k=\overline{1,n}$. If $n\neq 0$, in the same way, we have

$$k(z) = \frac{\overline{a}_1-\overline{\lambda}}{z-a_1}\left(c_0\frac{z-b_1}{z-b_0} + c_1 + c_2\frac{z-b_1}{z-b_2} + \ldots + c_n\frac{z-b_1}{z-b_n}\right)\,, \quad z\in C_1$$

and $c_k=0$, $k\neq 1$. Hence, $c_k=0$, $k=\overline{0,n}$ and $k(z)\equiv 0$, $z\in C$. This contradicts our assumption $w(z)\not\equiv 0$. Thus, $n=0$, $k(z)=\frac{C}{z}$, $|z|=1$ and therefore $w(z)\equiv const$. □

In the context of the Smirnov spaces $E^2(G_\pm)$, the self-adjointness of the projections $P_\pm$ is equivalent to the orthogonality $E^2(G_+)\perp E^2(G_-)$ (for the definition of $E^2(G_\pm)$, see [6]). Recall that $\operatorname{Ran}P_\pm = E^2(G_\pm)$ and $\operatorname{Ker}P_\pm = E^2(G_\mp)$.

Corollary 1. *If* $E^2(G_+) \perp E^2(G_-)$ *with respect to the inner product* $(\cdot,\cdot)_{L^2(C,|dz|)}$, *then the curve* C *is a circle.*

Recall that the operator $\mathcal{K}_S(\cdot,\lambda)$ are bounded in $L^2(C, w(z)|dz|)$ if and only if $(C, w^{1/2}) \in A_2$ (i.e., $w^{1/2}$ is a Mackenhoupt weight). Under this condition we evidently have $w \in L^1(C, |dz|)$) and therefore we can establish a desired extension of criterion of selfadjointness.

Corollary 2. *The bounded Cauchy singular integral operator* $\mathcal{K}_S(\cdot,\lambda)$ *is self-adjoint in the Hilbert space* $L^2(C, w(z)|dz|)$ *if and only if the curve* C *is a circle and* $w(z) \equiv const$.

Note that this extension to the case of multiply-connected domains can be obtained from the corresponding result for simple connected domains. However, we prefer to regard it as consequence of our main theorem mainly with the aim to present a new proof of Krupnik's result.

Remark. In [7] B.Sz.-Nagy and C.Foiaş employed the decomposition $L^2 = H^2 \oplus H^2_-$ to the construction of their functional model for contractions, where the spaces $H^2 = E^2(\mathbb{D})$, $H^2_- = E^2(\mathbb{D}_-)$ are Hardy's spaces. We emphasize that along with orthogonality there is an analyticity in both the domains $\mathbb{D}$ and $\mathbb{D}_-$, respectively. However, if we intend to extend the Sz.-Nagy-C.Foiaş functional model to arbitrary domains, we cannot keep simultaneously both these analyticities and orthogonality because of nonorthogonality of the decomposition $L^2(C) = E^2(G_+) + E^2(G_-)$.

The combination "analyticity only in G_+ plus orthogonality" is a mainstream of development in the multiply-connected case (see, e.g., [8] or [9]). In particular, the Riemann surface (=double of the planar domain G_+) is used therein and the authors have to deal with the "finite-rank defect" decomposition $L^2(C) = H^2_+ \oplus H^2_- \oplus \mathfrak{M}$, where $0 \neq \dim \mathfrak{M} < \infty$ and the subspace H^2_- corresponds to the duplicate of G_+ (and therefore we lose entirely geometrical information concerning the domain G_-). Another drawback of this approach is the use of uniformization technique or analytic vector bundles (with fairly large amount of algebraic geometry).

On the other hand, the nonorthogonal theory keeping both analyticities and free of the above drawbacks was developed recently in [10, 11]. Note that the duality with respect to the Cauchy pairing

$$< f, g >_C := \frac{1}{2\pi i} \int_C f(z) \cdot \overline{g(\bar{z})}\, dz\,, \quad f \in L^2(C),\ g \in L^2(\bar{C})$$

is a substitute for orthogonality in our approach. For instance, $E^2(G_\pm)^{<\perp>} = E^2(\overline{G_\pm})$ (see also [12]). Note also that linear similarity (instead of unitary equivalence) is a natural kind of equivalence in our theory.

References

1. David G., *L'integral de Cauchy sur les courbes rectifiables.* Prepublications Univ. Paris-Sud., Dept. Math., **82 T05**, 1982.
2. Böttcher A., Karlovich Yu., *Carleson Curves, Mackenhoupt Weights, and Toeplitz Operators*, Birkhauser, Basel, 1997.
3. Krupnik N., *The conditions of selfadjointness of the operator of singular integration.* Integral Equations Operator Theory **14** (1991), no. 5, 760—763.
4. Gohberg I., Krupnik N., *Linear Singular Integral Operators. Closed Countour, Continuous Coefficients*, Birkhauser, Basel, 1991.
5. Krupnik N., Markus A., Feldman I. *On the norm of projections in Hilbert space.* Funct. Anal. Appl. **23** (1989), no. 4, 88—89.
6. Goluzin G. M., *Geometric theory of functions of one complex variable*, Nauka, Moscow, 1966.
7. Szökefalvi-Nagy B., Foiaş C., *Harmonic analysis of operators on Hilbert space.* North-Holland, Amsterdam-London 1970.
8. Fedorov S. I., *On harmonic analysis in multiply connected domain and character-automorphic Hardy spaces.* Algebra i Analiz, **9** (1997), no.2, 192–239.
9. Ball J., Vinnikov V. *Hardy spaces on a finite bordered Riemann surface, multivariable operator model theory and Fourier analysis along a unimodular curve.* Oper. Theory Adv. Appl., **129**, Birkhauser, Basel, 2001, 37—56.
10. Tikhonov A. S., *Functional model and duality of spectral components for operators with continuous spectrum on a curve.* Algebra i Analiz, **14** (2002), no.4, 158–195.
11. Yakubovich D. V., *Linearly similar model of Szökefalvi-Nagy – Foias in a domain.* Algebra i Analiz, **15** (2003), no.2, 180–227.
12. Tumarkin G. C., Havinson S. Ja., *Mutual orthogonality of boundary values of certain classes of analytic functions in multiply connected domains*, Uspehi Mat. Nauk, **14** (1959), no. 3, 173–180.