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# A NEW CLASS OF PARABOLIC PROBLEMS CONNECTED WITH NEWTON'S POLYGON

## 1. Introduction

In this note two linearized problems of crystallization are discussed: the Stefan problem with Gibbs-Thomson correction and the Cahn-Hilliard equation with dynamic boundary condition. These two problems have common specific features: the equation in the interior of the domain is of rather simple structure and is parabolic in the sense of Petrovskii. The difficulties are connected with the boundary conditions which do not fit into the classical theory of parabolic problems.

The theory of parabolic (as well as elliptic and parameter-elliptic) problems even for scalar operators is deeply connected with the theory of mixed order system of pseudodifferential operators acting on the boundary. The matrix-symbol of this system is the so-called Lopatinskii matrix. In standard parabolic problems, the corresponding system is parabolic in the sense of Solonnikov [8]. We introduce a more general class of parabolic boundary value problems replacing systems parabolic in the sense of [8] by a more general class of N-parabolic systems studied by the second author in [10]. In the definition of these systems the notion of Newton's polygon plays a crucial role. It will be shown that the two mentioned examples from mathematical physics belong to this new class of boundary problems.

We now come to the formulation of the two problems under consideration. For simplicity of presentation, we restrict ourselves to the model case of the half-space $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ with boundary $\mathbb{R}^{n-1}$.

First, the linearized Stefan problem with Gibbs-Thompson correction is given by

$$\partial_t u(x,t) - \Delta u(x,t) = f(x,t) \quad (t > 0, x \in \mathbb{R}^n_+), \tag{1}$$

$$u(x',0,t) + \Delta' \sigma(x',t) = g(x',t) \quad (t > 0, x' \in \mathbb{R}^{n-1}), \tag{2}$$

$$\partial_n u(x',0,t) - \partial_t \sigma(x',t) = h(x',t) \quad (t > 0, x' \in \mathbb{R}^{n-1}), \tag{3}$$

$$u_0,\ \sigma(x',0) = \sigma_0. \tag{4}$$

Here $\Delta'$ is the Laplace operator on the boundary $\mathbb{R}^{n-1}$ and $\partial_n$ stands for the normal derivative.

In the case of zero initial conditions we can consider the problem on the whole time axis, supposing that

$$u(x,t) = f(x,t) = 0, \quad \sigma(x',t) = g(x',t) = h(x',t) = 0, \quad (t < 0).$$

The Stefan problem (1)–(4) was studied in a number of papers. Recently it was treated in detail by Escher-Prüss-Simonett in [5]. It was shown that in appropriate solution spaces this equation is uniquely solvable. Whereas in [5] the approach is semigroup based, we will see below that we can understand the structure of this boundary value problem and of the solution spaces in terms of the Newton polygon. Below we will show that the classical parabolicity condition is not satisfied for the Stefan problem (1)–(4).

Now let us come to the linearized Cahn-Hilliard equation with dynamic boundary conditions. It is given by

$$\partial_t u(x,t) + \Delta^2 u(x,t) = f(x,t) \quad (t > 0, x \in \mathbb{R}^n_+), \tag{5}$$

$$\partial_n \Delta u(x',0,t) = g(x',t) \quad (t > 0, x' \in \mathbb{R}^{n-1}), \tag{6}$$

$$\partial_t u(x',0,t) + \partial_n u(x',0,t) - \Delta' u(x',0,t) = h(x',t) \quad (t > 0, x' \in \mathbb{R}^{n-1}), \tag{7}$$

$$u_0. \tag{8}$$

The solvability of this problem in appropriate Sobolev spaces was recently investigated by Prüss-Racke-Zheng in [6]. Again the method was based on a semigroup approach. We will see below that also the Cahn-Hilliard equation (5)–(8) fits in the context of parabolic problems connected with the Newton polygon.

Below we will denote the co-variable to $t$ by $\tau$ and the co-variables to $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^{n-1}$ by $\xi \in \mathbb{R}^n$ and $\xi' := (\xi_1, \ldots, \xi_{n-1})$. In the following, $C$ stands for an unspecified constant which may vary from one appearance to the other but which is independent of the free variables. The notion $f \approx g$ means that there exists a positive constant $C$ for which $C^{-1} f \le g \le Cf$.

## 2. General parabolic problems and the Lopatinskii matrix

The examples above are particular cases of more general problems (see also [7]). In the interior of the domain we will consider an equation of the form

$$A(D_t, D_x)u(x,t) = f(x,t) \quad (x \in \mathbb{R}^n_+, t \in \mathbb{R}), \tag{9}$$
$$u(x,t) = f(x,t) = 0 \quad (t < 0).$$

We will assume that $A$ is $2b$-parabolic in the sense of Petrovskii and we will denote the order of $A$ by $2m$. On the boundary we have $\kappa$ additional functions $\sigma_1, \ldots, \sigma_\kappa$. Consequently, we need $m + \kappa$ boundary conditions

$$B_j(D_t, D_x)u(x',t) + \sum_{k=1}^{\kappa} C_{jk}(D_{x'})\sigma_k = g_j(x',t) \quad (j = 1, \ldots, m+\kappa). \tag{10}$$

The $2b$-parabolicity condition on $A$ means that

$$A_0(\tau,\xi) \neq 0 \quad (|\xi|^{2b} + |\tau| = 1,\ \operatorname{Im}\tau \leq 0), \tag{11}$$

where $A_0$ is the principal part of $A$. Here for the definition of the principal part we have to assign the weight $2b$ to the co-variable $\tau$. For simplicity we will suppose that our operators have constant coefficients and that $A_0$ coincides with $A$.

To define the Lopatinskii matrix and further to define parabolic problems, we have to introduce the factorization of $A(\tau,\xi)$ in the form

$$A(\tau,\xi',\xi_n) = A^+(\tau,\xi',\xi_n)A^-(\tau,\xi',\xi_n)$$

with

$$A^+(\tau,\xi',z) := \prod_{j=1}^{m}(z - z_j^+(\tau,\xi')) = z^m + \sum_{\ell=1}^{m} a_\ell(\tau,\xi')z^{m-\ell}.$$

Here $z_j^+(\tau,\xi')$ are the roots of $A(\tau,\xi',\cdot)$ with positive imaginary part.

For $\ell \geq m$ we write

$$z^\ell \equiv \sum_{k=1}^{m} \gamma_{\ell k}(\tau,\xi')z^{k-1} \mod A^+(\tau,\xi',z).$$

Replacing in the symbols $B_j(\tau,\xi,z)$ every power $z^\ell$ with $\ell \geq m$ by their remainder modulo $A^+(\tau,\xi',z)$, we get

$$B_j(\tau,\xi',z) \equiv \tilde{B}_j(\tau,\xi',z) \mod A^+(\tau,\xi',z)$$

with

$$\tilde{B}_j(\tau,\xi',z) = \sum_{k=1}^{m} b_{jk}(\tau,\xi')z^{k-1}.$$

Now we will define the Lopatinskii matrix for the problem (9). We set

$$L(\tau,\xi') = \big(L_{jk}(\tau,\xi')\big)_{j,k=1,\dots,m+\kappa}$$

with

$$L_{jk}(\tau,\xi') := \begin{cases} b_{jk}(\tau,\xi'), & j = 1,\dots,m+\kappa; \quad k = 1,\dots,m; \\ C_{j,k-m}(\tau,\xi'), & j = 1,\dots,m+\kappa; \quad k = m+1,\dots,m+\kappa. \end{cases}$$

Although the elements of the Lopatinskii matrix are algebraic functions we can define their orders. Indeed, as the symbol $A(\tau,\xi',z)$ is quasi-homogeneous in $(\tau,\xi',z)$ (where $\tau$ has weight $2b$), the same is true for its roots. Therefore, we have

$$|z_j^+(\tau,\xi')| \approx |\tau|^{\frac{1}{2b}} + |\xi'|, \quad \operatorname{Im} z_j^+(\tau,\xi') \geq C(|\tau|^{\frac{1}{2b}} + |\xi'|)$$

for $j = 1,\dots,m$.

It follows from the definition of $A^+$ that $a_\ell(\tau,\xi')$ are homogeneous polynomials of the roots $z_1^+,\dots,z_m^+$ of degree $\ell$. From the rule of division of polynomials it follows that the coefficients $\gamma_{\ell k}(\tau,\xi')$ are homogeneous polynomials of $z_1^+,\dots,z_m^+$ of degree $\ell+1-k$.

We will not suppose, in principle, that the polynomials $B_j(\tau,\xi)$ and $C_{jk}(\tau,\xi)$ are quasi-homogeneous. From the above construction we see that the functions $b_{jk}(\tau,\xi')$ are polynomials in $(\tau,\xi')$ and the roots $z_1^+,\dots,z_m^+$. Posing $\operatorname{ord}_{2b} z_j^+ = 1$, we will have

$$\operatorname{ord} b_{jk} \le m_j + 1 - k \quad (j = 1,\dots,m+\kappa,\ k = 1,\dots,m).$$

Here we have set $m_j := \operatorname{ord} B_j(\tau,\xi)$. From this we see that the orders of all entries of $L(\tau,\xi')$ are well-defined.

**Remark 1.** We add some remarks on the meaning of the Lopatinskii matrix. For $u$ belonging to the $L_2$-Sobolev space $H^{2m}(\mathbb{R}^n_+)$ we define the trace

$$\gamma_\ell u := \partial_n^\ell u|_{\mathbb{R}^{n-1}} \in H^{2m-\ell-\frac{1}{2}}(\mathbb{R}^{n-1}), \quad \ell = 0,\dots,2m-1.$$

Let $u$ be a solution of the equation with constant coefficients

$$A(D_t, D_x)u(x,t) = 0 \quad (x \in \mathbb{R}^n_+).$$

Then the Lopatinskii matrix is the symbol of the operator $L(D_t, D_{x'})$ mapping the vector of traces to the vector of the right-hand sides of (10)

$$L(D_t D_{x'}) : (\gamma_0 u,\ \gamma_1 u,\ \cdots,\gamma_{m-1} u,\sigma_1,\ \cdots,\sigma_\kappa) \to (g_1,\dots,g_{m+\kappa}).$$

In the case of variable coefficients the Lopatinskii matrix can be treated as the symbol of pseudofifferential operator with the same properties.

## 3. The parabolicity condition for the boundary value problem

For parabolic problems the analog of the Shapiro–Lopatinskii condition from elliptic theory was introduced in the papers of Eidelman (see [3]) and Solonnikov (see [8]). Agranovich and Vishik [2] studied elliptic problems with parameter and introduced the parameter-ellipticity condition which formally coincides with Eidelman-Solonnikov condition. The parameter-ellipticity condition was independently introduced by Agmon [1]. In the context of problems with parameter we shall call it Agmon–Agranovich–Vishik (AAV) condition. For scalar operators it can be formulated in terms of the principal part of the Lopatinskii matrix which we now define.

If we pose

$$s_j := m_j + 1 - m \quad (j = 1,\dots,m+\kappa), \quad t_k := m - k \quad (k = 1,\dots,m)$$

we have $\operatorname{ord} b_{jk} \le s_j + t_k$. For the operators $C_{jk}$ we set

$$t_k := \max\{\operatorname{ord} C_{j,k-m} - s_j : j = 1,\dots,m+\kappa\} \quad (k = m+1,\dots,m+\kappa).$$

In this way we obtain

$$\operatorname{ord} L_{jk}(\tau,\xi') \le s_j + t_k \quad (j,k = 1,\dots,m+\kappa).$$

Let $\pi_{2b}L_{jk}(\tau,\xi')$ be the principal part of $L_{jk}(\xi',\tau)$ with the weight of $\tau$ being $2b$. Then the principal part of the Lopatinskii matrix is defined as $L^0(\tau,\xi') = (L^0_{jk}(\tau,\xi'))_{j,k=1,\dots,m+\kappa}$ with

$$L^0_{jk}(\tau,\xi') := \begin{cases} \pi_{2b}L_{jk}(\tau,\xi') & \text{if } \operatorname{ord}_{2b} L_{jk} = s_j + t_k, \\ 0 & \text{if } \operatorname{ord}_{2b} L_{jk} < s_j + t_k. \end{cases}$$

**Definition 1.** We say that problem (9)–(10) satisfies the parabolicity or AAV condition if

(i) $\det L^0(\tau,\xi') = \pi_{2b}\det L(\tau,\xi')$, i.e. we have $\operatorname{ord}_{2b}\det L(\tau,\xi') = \sum_{j=1}^{m+\kappa}(s_j+t_j)$.

(ii) For all $\xi'$ and $\tau$ with $|\tau| + |\xi'| > 0$ and $\operatorname{Im}\tau \le 0$ we have

$$\det L^0(\tau,\xi') \neq 0.$$

In other words, the matrix $L(\tau,\xi')$ is parabolic in the sense of Solonnikov [8].

We will see now that neither the Stefan problem with Gibbs-Thomson correction nor the Cahn-Hilliard equation with dynamic boundary condition satisfy the condition (ii) of Definition 1.

Let us start with the Stefan problem (1)–(4). Using the factorization of the symbol of the heat equation

$$i\tau + |\xi'|^2 + \xi_n^2 = (\xi_n - i\sqrt{|\xi'|^2 + i\tau})(\xi_n + i\sqrt{|\xi'|^2 + i\tau}),$$

we calculate the Lopatinskii matrix

$$L(\tau,\xi') = \begin{pmatrix} 1 & -|\xi|^2 \\ -\sqrt{|\xi'|^2+i\tau} & -i\tau \end{pmatrix}. \tag{12}$$

According to the definition above, $s_1 = 0$, $s_2 = 1$, $t_1 = 0$, $t_2 = 2$, and the principal part of $L(\tau,\xi')$ is given by

$$L^0(\tau,\xi') = \begin{pmatrix} 1 & -|\xi|^2 \\ -\sqrt{|\xi'|^2+i\tau} & 0 \end{pmatrix}.$$

Condition (i) of Definition 1 is satisfied, but condition (ii) is violated because

$$\det L^0(\tau,\xi') = -|\xi'|^2\sqrt{|\xi'|^2+i\tau} = 0 \text{ for } |\xi'| = 0,\ |\tau| > 0.$$

Now let us consider the Cahn-Hilliard equation (5)–(8). We factorize $i\tau + (|\xi'|^2+\xi_n^2)^2 = A^+(\tau,\xi',\xi_n)\cdot A^-(\tau,\xi',\xi_n)$ with

$$A^+(\tau,\xi',\xi_n) = (\xi_n - z_1(\tau,\xi'))\cdot(\xi_n - z_2(\tau,\xi')).$$

The Lopatinskii matrix is given by

$$L(\tau,\xi') = \begin{pmatrix} iz_1z_2(z_1+z_2) & -i(z_1^2+z_2^2+z_1z_2+|\xi'|^2) \\ i\tau + |\xi'|^2 & i \end{pmatrix}. \tag{13}$$

In this case $\tau$ has weight 4 and $m_1 = 3, m_2 = 4$. Thus, $s_1 = 2, s_2 = 3, t_1 = 1, t_2 = 0$ and the principal part of $L$ is equal to

$$L^0(\tau, \xi') = \begin{pmatrix} iz_1z_2(z_1 + z_2) & -i(z_1^2 + z_2^2 + z_1z_2 + |\xi'|^2) \\ i\tau & 0 \end{pmatrix}.$$

Obviously, we have $\det L^0(\tau, \xi') = 0$ for $\tau = 0$ and arbitrary $\xi'$. Again the parabolicity condition is not satisfied.

## 4. Parabolic systems connected with the Newton polygon

In this section we introduce a more general class of parabolic problems such that the examples above belong to it. The main idea is to replace the traditional principal part of the Lopatinskii matrix by a principal part connected with the Newton polygon of $\det L$. We shall need some definitions and results from [4] and [10].

Consider a polynomial

$$P(\tau, \xi) = \sum_{\alpha, j} p_{\alpha j} \xi^\alpha \tau^j$$

in the variables $\xi \in \mathbb{R}^n$ and $\tau \in \mathbb{C}$. Then the Newton polygon $N(P)$ of the polynomial $P$ is defined as the convex hull of all points $(|\alpha|, j)$ for which $p_{\alpha j} \neq 0$, the projections of all these points to the coordinate axes and the origin. The following definition is taken from [4].

**Definition 2.** The polynomial $P(\tau, \xi)$ is called N-parabolic if the following conditions hold.

(i) The Newton polygon $N(P)$ has no edge parallel to the coordinate axes (except the trivial ones).

(ii) There exists a $\tau_0 < 0$ such that the estimate

$$|P(\tau, \xi)| \geq C \sum_{(i,j) \in N(P) \cap \mathbb{Z}^2} |\xi|^i |\tau|^j \quad (\xi \in \mathbb{R}^n, \operatorname{Im} \tau \leq \tau_0)$$

holds. Here the sum runs over all integer points belonging to the Newton polygon.

N-parabolic polynomials can be included in the class of so-called N-parabolic matrices (see [10]). For this, we consider a polynomial matrix

$$P(\tau, \xi) = \big(P_{jk}(\tau, \xi)\big)_{j,k=1,\dots,N}$$

and write the determinant of $P$ in the form

$$\det P(\tau, \xi) = \sum_\gamma \pm P_{1,\gamma(1)} \cdots P_{N,\gamma(N)}$$

where $\gamma$ runs through all permutations of the set $\{1, 2, \dots, N\}$. Assigning weight $\rho$ to the variable $\tau$ and weight 1 to the variables $\xi$ we define

$$R(\rho) := \max_\gamma \big( \operatorname{ord}_\rho P_{1,\gamma(1)} + \cdots + \operatorname{ord}_\rho P_{N,\gamma(N)} \big).$$

**Definition 3.** a) The matrix $P(\tau,\xi)$ is called totally non-degenerate if for every $\rho > 0$ we have

$$R(\rho) = \operatorname{ord}_\rho \det P(\tau,\xi).$$

b) The matrix $P(\tau,\xi)$ is called N-parabolic if the following conditions hold.

(i) $P(\tau,\xi)$ is totally non-degenerate.

(ii) The determinant $\det P(\tau,\xi)$ is N-parabolic.

For N-parabolic matrices $P(\tau,\xi)$ it is possible to define the principal part $P_\rho(\tau,\xi)$ for every fixed weight $\rho > 0$. As it was noted in [10], according to [9] for every $\rho$ there exist real numbers $s_j(\rho)$ and $t_k(\rho)$ $(j,k = 1,\dots,N)$ for which

$$\operatorname{ord}_\rho P_{jk}(\tau,\xi) \le s_j(\rho) + t_k(\rho), \quad j,k = 1,\dots,N,$$

$$\operatorname{ord}_\rho \det P = \sum_{j=1}^{N} (s_j(\rho) + t_j(\rho)).$$

The principal part $P_\rho(\tau,\xi)$ is defined in the standard way.

The matrix $P(\tau,\xi)$ is called *positively totally non-degenerate* if the above functions $s_i(\rho), t_i(\rho), i = 1,\dots,N$ can be chosen nonnegative. Replacing in Definition 3 b) totally non-degenerate matrices by positively totally non-degenerate, we define positively N-parabolic matrices.

We return to the problem (9). Formally the above definitions cannot be applied to the Lopatinskii matrix, because its elements are not polynomials but algebraic functions of special structure. In fact, Definition 2 and Definition 3 use only the possibility to calculate $\operatorname{ord}_\rho P_{ij}$ for each element $P_{ij}$ and each $\rho$.

In the case of the Lopatinskii matrix we can do this. Indeed, as it was mentioned above, the elements of this matrix are polynomials in $(\tau,\xi')$ and the roots $z_1^+,\dots,z_m^+$. Setting $\operatorname{ord}_\rho z_j^+ = \rho/2b$ for $\rho \ge 2b$ and $\operatorname{ord}_\rho z_j^+ = 1$ for $\rho \le 2b$ we define $\operatorname{ord}_\rho$ for each element of the Lopatinskii matrix. So we can give

**Definition 4.** The problem (9)–(10) is called N-parabolic if its Lopatinskii matrix $L(\tau,\xi)$ is N-parabolic in the sense of Definition 3.

One of the main results of [10] is devoted to the construction of $L_2$-Sobolev vector spaces with weight functions $\mu_j(\tau,\xi), \nu_j(\tau,\xi), j = 1,\dots,N$ in which the operator $P(D_t, D_x)$ is invertible. Now we formulate this result.

First of all we define the corresponding class of spaces. For this denote by $\mathcal{F}$ the set of all positive functions $\chi : \mathbb{R}^n \times \mathbb{C}_- \to (0,\infty)$ holomorphic in $\mathbb{C}_-$ for which

$$\chi(\xi_1,\tau_1)\chi^{-1}(\xi_2,\tau_2) \le C(1 + |\tau_1-\tau_2| + |\xi_1-\xi_2|)^{M(\mu)} \quad (\tau_1,\tau_2 \in \mathbb{C}_-,\ \xi_1,\xi_2 \in \mathbb{R}^n)$$

C. Let $\chi$ be a weight function belonging to the class $\mathcal{F}$ defined above. Then for $\gamma \in \mathbb{R}$ the norm $\|\cdot, H^{\chi}_{[\gamma]}(\mathbb{R}^{n+1})\|$ is defined by

$$\|u, H^{\chi}_{[\gamma]}(\mathbb{R}^{n+1})\| := \Big( \int\limits_{\operatorname{Im}\tau=\gamma} \int\limits_{\mathbb{R}^n} \chi(\xi,\tau)^2 |\hat{u}(\xi,\tau)|^2 d\xi d\tau \Big)^{1/2}.$$

The Sobolev space $H^{\chi}_{[\gamma]}(\mathbb{R}^{n+1})$ is the set of all tempered distributions for which this norm is finite. In the case $\chi(\xi,\tau) \approx 1 + |\tau|^r + |\xi|^s$, we will also write $H^{r,s}_{[\gamma]}(\mathbb{R}^{n+1})$ instead of $H^{\chi}_{[\gamma]}(\mathbb{R}^{n+1})$.

By $H^{\chi}_{[\gamma]}(\mathbb{R}^{n+1})_+$ we denote the subspace of all distributions in $H^{\chi}_{[\gamma]}(\mathbb{R}^{n+1})$ whose support is contained in $\mathbb{R}^n \times [0,\infty)$. By $H^{\chi}_{[\gamma]}(\mathbb{R}^n)_+$ we denote the corresponding space for functions defined on the hyperplane $\{x_n = 0, (x',t) \in \mathbb{R}^n\}$. The following result is proved in [10].

**Proposition 1.** Let $P(\tau,\xi) = \big(P_{jk}(\tau,\xi)\big)_{j,k=1,\dots,N}$ be positively N-parabolic. Then there exist functions $\mu_j(\tau,\xi)$, $j = 1,\dots,N$, and $\nu_k(\tau,\xi)$, $k = 1,\dots,N$, belonging to $\mathcal{F}$ such that

$$P_{jk}(\tau,\xi) \le C\mu_j(\tau,\xi)\cdot\nu_k(\tau,\xi)$$

and

$$|\det P(\tau,\xi)| \ge C \prod_{j=1}^{N} \mu_j(\tau,\xi)\cdot\nu_j(\tau,\xi) \quad (\operatorname{Im}\tau \le \tau_0).$$

The proof of Proposition 1 contains the algorithm of computing the weight-functions $\mu_1,\dots,\nu_N$. Using Plancherel's theorem, it is possible to obtain information about the invertibility of the operator $P(D_t, D_x)$. So we have the following result

**Theorem 1.** *Suppose the Lopatinskii matrix $L(\tau,\xi)$ is positively N-parabolic. Then for $P(\tau,\xi) = L(\tau,\xi)$ the weights from Proposition 1 exist, and for each $\chi(\tau,\xi) \in \mathcal{F}$ the isomorphism between the weighted Sobolev spaces*

$$L(D_t,D_x)\colon \prod_{j=1}^{N} H^{\chi\cdot\mu_j}_{[\gamma]}(\mathbb{R}^n)_+ \to \prod_{k=1}^{N} H^{\chi\cdot\nu_k^{-1}}_{[\gamma]}(\mathbb{R}^n)_+$$

*holds provided $-\gamma$ is large enough, i. e. $\gamma \le \gamma_0(\chi)$.*

However, when trying to apply Theorem 1 to the initial-boundary value problem (9)–(10) we meet one principle difficulty. It is natural to write the solution $u$ in the form $u = u_0 + u_1$ where $u_0$ is a solution of the homogeneous Dirichlet problem

$$\begin{aligned} A(D_t,D_x)u_0(x,t) &= f(x,t) && \text{in } \mathbb{R}^n_+ \times [0,\infty), \\ \partial_n^{k-1} u_0(x',t) &= 0 && \text{in } \mathbb{R}^{n-1} \times [0,\infty), \quad k = 1,\dots,m. \end{aligned} \tag{14}$$

This equation can be treated with standard parabolic theory. The second function $u_1$ is the solution of the following problem with inhomogeneous boundary data.

$$\begin{aligned} A(D_t, D_x)u_1(x,t) &= 0 && \text{in } \mathbb{R}^n_+ \times [0,\infty), \\ B_j(D_t, D_x)u_1(x',t) + \sum_{k=1}^{\kappa} C_{jk}(D_{x'})\sigma_k &= \tilde{g}_j(x',t) && \text{in } \mathbb{R}^{n-1} \times [0,\infty) \end{aligned} \tag{15}$$

where

$$\tilde{g}_j(x',t) = g_j(x',t) - B_j(D_t, D_x)u_0(x',t), \quad j = 1, \dots, m+\kappa.$$

Starting with $f \in H^0_{[\gamma]}(\mathbb{R}^{n+1})_+$, we obtain $u_0 \in H^{1,2m}_{[\gamma]}(\mathbb{R}^{n+1})_+$. For the sum $u = u_0 + u_1$ to belong to this space its Dirichlet data must belong to the vector-space $\prod_{j=0}^{m-1} H^{1-j/4m, 2m-j/2}_{[\gamma]}(\mathbb{R}^n)_+$. On the other side, applying Theorem 1 to problem (15) we obtain that the Dirichlet data belongs to $\prod_{j=1}^{m} H^{\chi\mu_j}_{[\gamma]}(\mathbb{R}^n)_+$ provided the right-hand sides $\tilde{g}_j(x',t)$ belong to $H^{\chi/\nu_j}_{[\gamma]}(\mathbb{R}^n)_+$ with some $\chi$. The possibility to choose $\chi$ such that all these conditions will be compatible demands additional investigation. But in the case of the above examples from crystallization such choice is possible and will lead to two-sided estimates.

## 5. Stefan problem with Gibbs-Thomson correction

For the Stefan problem (1)–(4), the Lopatinskii matrix is given by (12), and its determinant equals

$$\det L(\tau, \xi') = -(|\xi'|^2 \sqrt{|\xi'|^2 + i\tau} + i\tau).$$

Note that for $\operatorname{Im}\tau \leq 0$ we have $|\sqrt{|\xi'|^2 + i\tau}| \approx |\xi'| + |\tau|^{1/2}$. Therefore, the Newton polygon of $\det L(\tau, \xi')$ has the vertices $(0,0), (0,1), (2, \frac{1}{2})$ and $(3,0)$ (see Figure 1).

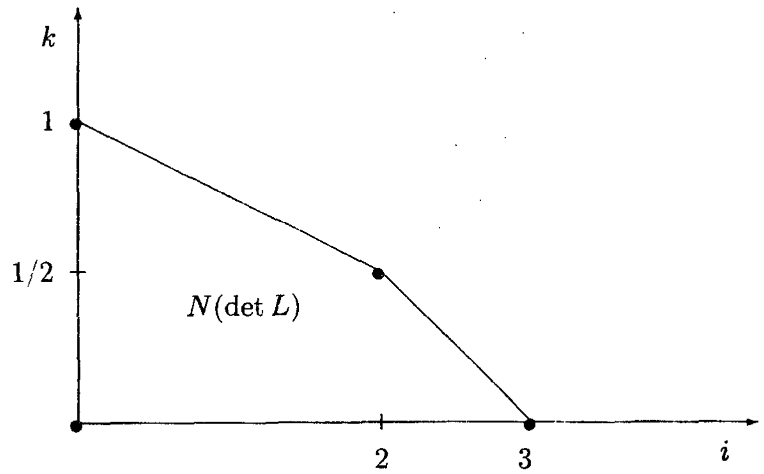


Рис. 1. The Newton polygon for the Stefan problem

For every $\rho > 0$, we define $s_j(\rho)$ and $t_k(\rho)$ $(j, k = 1, 2)$ by $s_1(\rho) := 0$, $t_1(\rho) := 0$ and

$$s_2(\rho) := \begin{cases} \frac{\rho}{2}, & \rho \geq 2, \\ 1, & \rho \leq 2, \end{cases} \qquad t_2(\rho) := \begin{cases} \frac{\rho}{2}, & \rho \geq 4, \\ 2, & \rho \leq 4. \end{cases}$$

Then

$$\operatorname{ord}_\rho L_{jk}(\tau, \xi') \leq s_j(\rho) + t_k(\rho) \quad (j, k = 1, 2)$$

and

$$\sum_{j=1}^{2} (s_j(\rho) + t_j(\rho)) = \begin{cases} \rho, & \rho \geq 4, \\ 2 + \frac{\rho}{2}, & 4 \geq \rho \geq 2, \\ 3, & \rho \leq 2. \end{cases}$$

For all $\rho$, the right-hand side of the last equality coincides with $\operatorname{ord}_\rho \det L(\tau, \xi')$. From this it can be seen that the Stefan problem (1)–(4) is N-parabolic.

Applying the general technique of [10] we show that the weight functions

$$\nu_1(\tau, \xi') = \mu_2(\tau, \xi') = 1,$$
$$\nu_2(\tau, \xi') = 1 + |\xi'| + |\tau|^{1/2}, \quad \mu_2(\tau, \xi') = 1 + |\xi'|^2 + |\tau|^{1/2}$$

satisfy the conditions of Proposition 1.

According to Theorem 1, the system of pseudodifferential equations

$$L(D_{x'}, D_t) \begin{pmatrix} v(x', t) \\ \sigma(x', t) \end{pmatrix} = \begin{pmatrix} g(x', t) \\ h(x', t) \end{pmatrix}.$$

for every $(f, g) \in H^{\chi}_{[\gamma]}(\mathbb{R}^n)_+ \times H^{\chi \cdot \nu_2^{-1}}_{[\gamma]}(\mathbb{R}^n)_+$ has a unique solution $(v, \sigma) \in H^{\chi}_{[\gamma]}(\mathbb{R}^n)_+ \times H^{\chi \cdot \mu_2}_{[\gamma]}(\mathbb{R}^n)_+$ for large enough $-\gamma$ (remind that $\mu_1 = \nu_1 = 1$). The a priori estimate

$$\|\chi(D_{x'}, D_t)v\|_{[\gamma]} + \|(\chi \cdot \mu_2)(D_{x'}, D_t)\sigma\|_{[\gamma]} \leq C\Big(\|\chi(D_{x'}, D_t)g\|_{[\gamma]} + \|(\chi \cdot \nu_2^{-1})(D_{x'}, D_t)h\|_{[\gamma]}\Big) \tag{16}$$

holds.

Now we estimate the solution of the linearized Stefan problem, following the plan indicated at the end of the previous section. We pose $u = u_0 + u_1$, where the function $u_0$ is a solution of the heat equation with zero Dirichlet boundary condition:

$$\partial_t u_0(x, t) - \Delta u_0(x, t) = f(x, t), \quad (x, t) \in \mathbb{R}^{n+1},$$
$$u_0(x', t) = 0, \quad (x', t) \in \mathbb{R}^n,$$
$$u_0(x, t) = f(x, t) = 0, \quad t < 0.$$

For $f \in H^0_{[\gamma]}(\mathbb{R}^{n+1})_+$ we have $u_0 \in H^{1,2}_{[\gamma]}(\mathbb{R}^{n+1})_+$ and, according to the trace theorem, $(\partial_n u_0)(x', t) \in H^{1/4,1/2}_{[\gamma]}(\mathbb{R}^n)_+$. For $u_1$ and $\sigma$ we obtain the problem

$$\partial_t u_1(x, t) - \Delta u_1(x, t) = 0,$$
$$u_1(x', t) + \Delta' \sigma(x', t) = \tilde{g}(x', t) := g(x', t),$$

$$\partial_n u_1(x',t) - \partial_t \sigma(x',t) = \tilde{h}(x',t) := h(x',t) - \partial_n u_0(x',t).$$

Now we make the choice $\chi(\tau,\xi) := 1 + |\tau|^{3/4} + |\xi'|^{3/2}$. In this case

$$\lambda(\tau,\xi') := (\chi \cdot \mu_2)(\tau,\xi') \approx 1 + |\tau|^{5/4} + |\xi'|^{7/2} + |\tau|^{3/4}|\xi'|^2,$$
$$(\chi \cdot \nu_2^{-1})(\tau,\xi') \approx 1 + |\tau|^{1/4} + |\xi'|^{1/2},$$

and inequality (16) transforms into

$$\|u, H^{3/4,3/2}_{[\gamma]}(\mathbb{R}^n)\| + \|\sigma, H^{\lambda}_{[\gamma]}(\mathbb{R}^n)\| \le C\Big(\|g, H^{3/4,3/2}_{[\gamma]}(\mathbb{R}^n) + \|h, H^{1/4,1/2}_{[\gamma]}(\mathbb{R}^n)\| + \|\partial_n u_0, H^{1/4,1/2}_{[\gamma]}(\mathbb{R}^n)\|\Big). \tag{17}$$

According to trace theorems and the estimate of the Dirichlet problem for the heat equation, the last term on the right-hand side can be estimated by

$$C\|u_0, H^{1,2}_{[\gamma]}(\mathbb{R}^{n+1})\| \le C\|f, H^0_{[\gamma]}(\mathbb{R}^{n+1})\|.$$

On the other side, according to the standard estimate for the heat equation

$$\|u, H^{1,2}_{[\gamma]}(\mathbb{R}^{n+1})\| \le C(\|f, H^0_{[\gamma]}\| + \|u, H^{1/4,1/2}_{[\gamma]}(\mathbb{R}^n)\|).$$

As a result we obtain the two-sided estimate for the linearized Stefan problem with Gibbs-Thomson correction

$$\|u, H^{1,2}_{[\gamma]}(\mathbb{R}^{n+1})\| + \|\sigma, H^{\lambda}_{[\gamma]}(\mathbb{R}^n)\| \le C\Big(\|f, H^0_{[\gamma]}\| + \|g, H^{3/4,3/2}_{[\gamma]}(\mathbb{R}^n)\| + \|h, H^{1/4,1/2}_{[\gamma]}(\mathbb{R}^n)\|\Big). \tag{18}$$

This estimate coincides with the estimates of [5] for the case $p = 2$.

## 6. Cahn-Hilliard equation

Let us consider the Cahn-Hilliard equation with dynamic boundary condition (5)–(6).

As we have already noted, the Lopatinskii matrix for the Cahn-Hilliard equation is given by (13) and its determinant equals

$$L(\tau,\xi') = -(i\tau + |\xi'|^2)(z_1^2 + z_2^2 + z_1 z_2 + |\xi'|^2) - z_1 z_2(z_1 + z_2).$$

We will need an elementary lemma.

**Lemma 1.** *Let* $z_1 = z_1(\tau,\xi')$ *and* $z_2 = z_2(\tau,\xi')$ *be the roots of the equation*

$$(z^2 + |\xi'|^2)^2 + i\tau = 0$$

*with* $\operatorname{Im} z_1 > 0$, $\operatorname{Im} z_2 > 0$ *where* $\operatorname{Im}\tau \le 0$. *Then*

$$|z_1^2 + z_2^2 + z_1 z_2 + |\xi'|^2| \approx |\xi'|^2 + |\tau|^{1/2}.$$

>From this lemma we obtain for large $-\gamma$

$$|L(\tau,\xi')| \approx (1+|\tau|+|\xi'|^2)(1+|\tau|^{1/2}+|\xi'|^2) \approx 1+|\tau|^{3/2}+|\xi'|^2|\tau|+|\xi'|^4.$$

Therefore, the Newton polygon of $\det L(\tau,\xi')$ has the vertices $(0,0), (0,\frac{3}{2}), (2,1)$ and $(4,0)$ (see Figure 2).

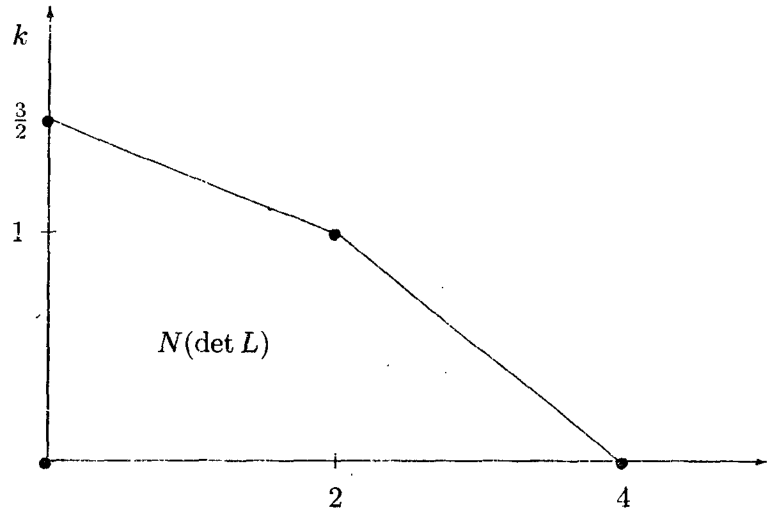


РИС. 2. The Newton polygon for the Cahn-Hilliard equation

For every $\rho > 0$, we define $s_j(\rho)$ and $t_k(\rho)$ $(j,k=1,2)$ by $s_2(\rho)=0$, $t_2(\rho)=0$, and

$$\begin{aligned} s_1(\rho)&=2,\ t_1(\rho)=2 &&\text{if } \rho\le 2,\\ s_1(\rho)&=2,\ t_1(\rho)=\rho &&\text{if } 2\le\rho\le 4,\\ s_1(\rho)&=\frac{\rho}{2},\ t_1(\rho)=\rho &&\text{if } \rho\ge 4. \end{aligned}$$

Then it can be checked that

$$\operatorname{ord}_\rho L_{jk}(\tau,\xi') \le s_j(\rho)+t_k(\rho) \quad (j,k=1,2)$$

and

$$\sum_{j=1}^{2}(s_j(\rho)+t_j(\rho)) = \operatorname{ord}_\rho L(\tau,\xi').$$

>From this it follows that the Cahn-Hilliard problem (5)–(8) is N-parabolic.

Applying the general technique of [10] we can show that the weight functions

$$\begin{aligned} &\nu_1(\tau,\xi') = 1+|\xi'|^2+|\tau|^{1/2}, \quad \mu_1(\tau,\xi')=1+|\xi'|^2+|\tau|,\\ &\nu_2(\tau,\xi')=\mu_2(\tau,\xi')=1 \end{aligned}$$

satisfy the conditions of Proposition 1.

As above, we represent the solution $u$ of our problem as $u = u_0 + u_1$, where the function $u_0 \in H^{1,4}_{[\gamma]}(\mathbb{R}^{n+1})$ satisfies the inhomogeneous Cahn-Hilliard equation and has zero Dirichlet data: $u_0(x', t) = (\partial_n u_0)(x', t) = 0$. We apply Theorem 1 to the difference $u_1 = u - u_0$ and obtain the estimate

$$||u, H^{\chi\cdot\mu_1}_{[\gamma]}(\mathbb{R}^n)|| + ||(\partial_n u, H^{\chi}_{[\gamma]}(\mathbb{R}^n)|| \le C\Big(||g - \partial_n \Delta u_0, H^{\chi\cdot\nu_1^{-1}}_{[\gamma]}(\mathbb{R}^n)|| + ||h, H^{\chi}_{[\gamma]}(\mathbb{R}^n)||\Big). \tag{19}$$

Now we take $\chi(\tau, \xi)$ such that the second term in the left-hand side will turn into the boundary norm of $\partial_n u$ for $u \in H^{1,4}_{[\gamma]}(\mathbb{R}^{n+1})$. It means that $\chi(\tau, \xi) := 1 + |\tau|^{5/8} + |\xi'|^{5/2}$. In this case

$$\lambda(\tau, \xi') := (\chi\mu_1)(\tau, \xi') \approx 1 + |\tau|^{13/8} + |\tau||\xi'|^{5/2} + |\xi'|^{9/2},$$
$$(\chi \cdot \nu_1^{-1})(\tau, \xi') \approx 1 + |\tau|^{1/8} + |\xi'|^{1/2},$$

and inequality (19) transforms into

$$||\gamma_0 u, H^{\lambda}_{[\gamma]}(\mathbb{R}^n)|| + ||\gamma_1 u, H^{3/4,3}_{[\gamma]}(\mathbb{R}^n)||$$
$$\le C\Big(||g, H^{1/8,1/2}_{[\gamma]}(\mathbb{R}^n)|| + ||h, H^{3/4,3}_{[\gamma]}(\mathbb{R}^n)|| + ||\gamma_0 \partial_n \Delta u_0, H^{1/8,1/2}_{[\gamma]}(\mathbb{R}^n)||\Big).$$

The last term on the right-hand side can be estimated by

$$C||u_0, H^{1,4}_{[\gamma]}(\mathbb{R}^{n+1})|| \le C||f, H^{0}_{[\gamma]}(\mathbb{R}^{n+1})||.$$

Taking advantage of the estimate

$$||u, H^{1,4}_{[\gamma]}(\mathbb{R}^{n+1})|| \le C\Big(||f, H^{0}_{[\gamma]}(\mathbb{R}^{n+1})|| + ||\gamma_0 u, H^{7/8,7/2}_{[\gamma]}(\mathbb{R}^n)|| + ||\gamma_1 u, H^{3/4,3}_{[\gamma]}(\mathbb{R}^n)||\Big),$$

we come to the final inequality

$$||\gamma_0 u, H^{\lambda}_{[\gamma]}(\mathbb{R}^n)|| + ||u, H^{1,4}_{[\gamma]}(\mathbb{R}^{n+1})||$$
$$\le C\big(||f, H^{0}_{[\gamma]}(\mathbb{R}^{n+1})|| + ||g, H^{1/8,1/2}_{[\gamma]}(\mathbb{R}^n)|| + ||h, H^{3/4,3}_{[\gamma]}(\mathbb{R}^n)||\big).$$

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