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REPRESENTATIONS OF *-ALGEBRAS ASSOCIATED WITH DYNKIN GRAPHS AND HORN'S PROBLEM

1. Introduction.

Let A, B, C – be complex Hermitian n by n matrices with the n-tuples of eigenvalues $\tau(A) = \{\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)\}$, $\tau(B) = \{\lambda_1(B) \geq \lambda_2(B) \geq \ldots \geq \lambda_n(B)\}$, $\tau(C) = \{\lambda_1(C) \geq \lambda_2(C) \geq \ldots \geq \lambda_n(C)\}$. The well know classical problem about the spectrum of sum of two Hermitian matrices (Horn' problem) is to describe the relations between $\tau(A), \tau(B), \tau(C)$ for the matrices A, B, C such that $A + B = C$. The recent solution to this problem (see [1, 2]) is given in terms of linear inequalities of the form

$$\sum_{i\in I} \lambda_i(A) + \sum_{j\in J} \lambda_j(B) \geq \sum_{k\in K} \lambda_k(C), \tag{1}$$

where I, J, K – certain subsets $\{1, \ldots, n\}$ of the same cardinality. Remark at this point that the number of necessary inequalities increases with n.

In present work we consider the following modification of the mentioned problem. Let us consider bounded Hermitian operators in separable Hilbert space H. Denote by $\sigma(X)$ the spectrum of the Hermitian operator X. Let M_1, M_2, M_3 be fixed closed subsets of in $\mathbb{R}_0 = \{x \in \mathbb{R} | x \geq 0\}$. We assume that each of M_k contains zero. Let us fix also $\gamma \in \mathbb{R}^+$. Consider the following problem: do there exist Hermitian operators A, B, C such that $\sigma(A) = M_1$, $\sigma(B) = M_2$, $\sigma(C) = M_3$ and $A + B + C = \gamma I_H$? (Further on we will refer to this problem as algebraic Horn's problem).

The essential difference with classical problem is that we do not fix the dimension of the H (it can also be infinite) and multiplicities of spectral points. It seems that solution to this problem could not be obtained from Horn's inequalities (1) since the number of necessary inequalities increases with n. Further on in the paper M_k will be finite.

Then this problem can be stated in terms of representations of *-algebras. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_l), \delta = (\delta_1, \delta_2, \ldots, \delta_m)$ be vectors with positive real strictly increasing coefficients. Let us define an algebra given by generators and relations (see [3, 4]):

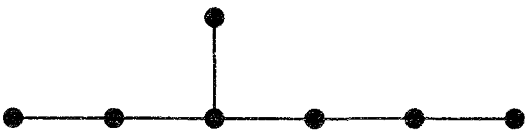
$$\mathcal{P}_{\alpha,\beta,\delta,\gamma} = \mathbb{C}\langle p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_l, s_1, s_2, \ldots, s_m | \\ p_i p_j = \delta_{ij} p_i, q_i q_j = \delta_{ij} q_i; s_i s_j = \delta_{ij} s_i; \\ \sum_{i=1}^{k} \alpha_i p_i + \sum_{j=1}^{l} \beta_j q_j + \sum_{d=1}^{m} \delta_d s_d = \gamma e\rangle.$$

Here e denotes the identity of the algebra, δ_{ij} – Kroneker symbol. This is a *-algebra if we require all generators be self-adjoint.

We will call *-representation π of the algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ non-degenerate if $\pi(p_i) \neq 0$, $\pi(q_j) \neq 0$, $\pi(s_d) \neq 0$, for all $1 \leq i \leq k$, $1 \leq j \leq l$, $1 \leq d \leq m$, and $\sum_{i=1}^{k} \pi(p_i) \neq I$, $\sum_{j=1}^{l} \pi(q_i) \neq I$, $\sum_{d=1}^{m} \pi(s_i) \neq I$. Define the set $T_{k,l,m} = \{(\alpha,\beta,\delta,\gamma) | \alpha \in \mathbb{R}^{+k}, \beta \in \mathbb{R}^{+l}, \delta \in \mathbb{R}^{+m}, \gamma \in \mathbb{R}^{+}\}$. Let $\mathrm{Rep}\,\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ denote the category of *-representations of the *-algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$, $\overline{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ – its full subcategory of of non-degenerate representations and $\widetilde{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ – the full subcategory of non-degenerate irreducible representations.

Denote $W_{k,l,m} = \{(\alpha,\beta,\delta,\gamma) \in T_{k,l,m} | \overline{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma} \neq \emptyset\}$ and $W^{irr}_{k,l,m} = \{(\alpha,\beta,\delta,\gamma) \in T_{k,l,m} | \widetilde{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma} \neq \emptyset\}$.

With integer vector (k,l,m) we associated a non-oriented graph G without loops (tree) with three rays of length k, l and m (not counting the root) correspondingly coming from a single root. For example, to a vector $(2,3,1)$ there correspond the following graph



Further on we will denote $W_{k,l,m}$, $T(k,l,m)$ and $W^{irr}_{k,l,m}$ as $W(G)$, $T(G)$, $W^{irr}(G)$ correspondingly, where G the tree constructed as above from the vector (k,l,m).

The algebraic Horn problem for the operators in Hilbert space then can be stated as follows: for the graph G describe the set $W(G)$. This description we reduce to the description of $W^{irr}(G')$ for each connected subgraph G' of the G and the problem of construction of $W(G)$ from the sets $W^{irr}(G')$ (combinatorial problem).

The solution of the problem is considerably less difficult in case G is a Dynkin graph. The algebras $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ associated with Dynkin graph (extended Dynkin graphs) is more manageable than in other cases. In particular, algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ is finite dimensional (has polynomial growth) if and only if G is a Dynkin graph (extended Dynkin graph) (see [4]).

In present work we will show that the problem of classification of *-representations of *-algebra$\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ associated with trees can be reformulated in terms of locally-scalar graph representations (see [5]). We will construct an equivalence between the category of *-representations of *-algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ and a certain full subcategory in the category of locally scalar representations of the corresponding graph (Theorem 1).

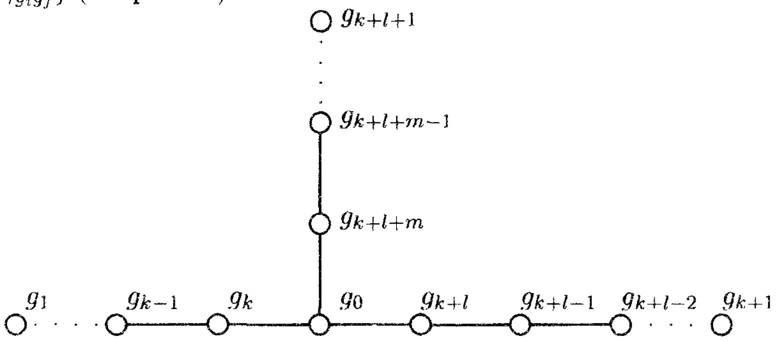
Irreducible locally-scalar representations of Dynkin graph are finite dimensional, the set of possible dimensions is finite (coincides with the set of positive roots of the Dynkin graph) and any represenation is a direct sum of irreducible ones (see [5]).

In the paper we will present the complete description of the set $W(G)$ for the algebra associated with Dynkin graph $\mathcal{D}_4$ (Theorem 2).

Let us remark that representations of the quivers (semi-stable representations) were also used in connection with Horn's problem in [6, 7].

2. Locally-scalar graph representations and representations of algebras generated by projections.

Let us consider a tree G with vertices $G_v\{g_i, i = 0, \dots, k + l + m\}$ and edges $G_e = \{\gamma_{g_i g_j}\}$ (see picture).



Further on we will stick to the definitions and notations used in [5]. Let us recall some of them. Let ε be the map which sends edge $\gamma \in G_e$ to the pair of vertices it connects. Denote by $\mathcal{H}$ the category of Hilbert spaces. Its objects are separable Hilbert spaces (finite dimensional as well as infinite dimensional) and its morphisms are bounded linear operators. The representation π of graph G in $\mathcal{H}$ assigns to each vertex $a \in G_v$ a Hilbert space $\pi(a) = H^a \in \mathrm{Ob}\mathcal{H}$ and to each edge $\gamma \in G_e$ with $\varepsilon(\gamma) = \{a, b\}$ assigns a pair of mutually adjoint linear operator $\pi(\gamma) = \{\Gamma_{ab}, \Gamma_{ba}\}$ where $\Gamma_{ab} : H^b \to H^a$. Denote by M_g the set of vertices connected with g by an edge. Consider an operator

$$A_g = \sum_{g_i \in M_g} \Gamma_{g g_i} \Gamma_{g_i g}.$$

A representation π of the graph G is called *locally-scalar* if all the operators A_g are scalar, i.e. $A_g = \alpha_g I_{H^g}$ where I_{H^g} denotes the identity operator in H^g. The function $f : G_v \to \mathbb{C}$, $g \mapsto \alpha_g$ will be called *character* of the locally-scalar graph representation π. If π is finite dimensional then function $d(g) = \dim \pi(g)$ will be called the dimension of π. The dimension is uniquely given by the representation but generally speaking is not uniquely determined by the character (function f is not uniquely defined for vertex g such that $d(g) = 0$).

The category $\mathrm{Rep}\,(G, \mathcal{H})$ of representations of the graph G in category $\mathcal{H}$ is defined as follows: a morphism from representation π to $\widetilde{\pi}$ in $\mathrm{Rep}\,(G, \mathcal{H})$ is a family of operators $\{C_g\}_{g \in G_v}$ such that for $\pi(\gamma) = \{\Gamma_{ab}, \Gamma_{ba}\}$, $\widetilde{\pi}(\gamma) = \{\widetilde{\Gamma}_{ab}, \widetilde{\Gamma}_{ba}\}$ ($\gamma \in G_e$, $\varepsilon(\gamma) = \{a, b\}$) the following diagrams are commutative:

$$\begin{array}{ccccccc}
\pi(a) & \xrightarrow{\Gamma_{ba}} & \pi(b) & \quad & \pi(b) & \longrightarrow & \pi(a) \\
{\scriptstyle C_a}\big\downarrow & & \big\downarrow & & {\scriptstyle C_a}\big\downarrow & & \big\downarrow{\scriptstyle C_a} \\
\widetilde{\pi}(a) & \xrightarrow{\widetilde{\Gamma}_{ba}} & \widetilde{\pi}(b) & \quad & \widetilde{\pi}(b) & \xrightarrow{\widetilde{\Gamma}_{ba}} & \widetilde{\pi}(a)
\end{array}$$

By $\mathrm{Rep}(G)$ we will mean the category of locally-scalar representations G (which is a full subcategory in $\mathrm{Rep}\,(G, \mathcal{H})$). Now we will define a functor Φ from the category

$\overline{\text{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ of non-degenerate representations in category $\mathcal{H}$ to the category Rep(G) of locally-scalar representations of the corresponding graph. Let $\pi \in \overline{\text{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ be a representation of $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ in Hilbert space H_0. We will denote by capital letters corresponding operators $P_i = \pi(p_i), Q_i = \pi(q_i), S_i = \pi(s_i)$. Let $H_{p_i} = \operatorname{Im} P_i, H_{q_i} = \operatorname{Im} Q_i, H_{s_i} = \operatorname{Im} S_i$. Denote by Γ_{p_i} the natural isometry from $H_{p_i} \to H_0$, Γ_{q_i} from $H_{q_i} \to H_0$ and by Γ_{s_i} from $H_{s_i} \to H_0$. Then

$$\Gamma^*_{p_i}\Gamma_{p_i} = I_{H_{p_i}}, \Gamma^*_{q_i}\Gamma_{q_i} = I_{H_{q_i}}, \Gamma^*_{s_i}\Gamma_{s_i} = I_{H_{s_i}},$$
$$\Gamma_{p_i}\Gamma^*_{p_i} = P_i, \Gamma_{q_i}\Gamma^*_{q_i} = Q_i, \Gamma_{s_i}\Gamma^*_{s_i} = S_i.$$

Let us construct a representation $\Pi \in \operatorname{Rep} G$. Put

$$\Pi(g_0) = H^{g_0} = H_0,$$
$$\Pi(g_k) = H^{g_k} = H_{p_1} \oplus H_{p_2} \oplus \ldots \oplus H_{p_k},$$
$$\Pi(g_{k-1}) = H^{g_{k-1}} = H_{p_1} \oplus H_{p_2} \oplus \ldots \oplus H_{p_{k-1}},$$
$$\Pi(g_{k-2}) = H^{g_{k-2}} = H_{p_2} \oplus H_{p_2} \oplus \ldots \oplus H_{p_{k-1}},$$
$$\ldots.$$

Here summands from left and from right are omitted in turn. Analogously, we define subspaces $\Pi(g_i)$ for $i = k+1, \ldots, k+l$ and for $i = k+l+1, \ldots, k+l+m$. In what follows we will denote by $\Gamma_{g_ig_j} : H_j \to H_i$ operator conjugated to $\Gamma_{g_jg_i} : H_i \to H_j$ (i.e. $\Gamma_{ij} = \Gamma^*_{ji}$). Let us define also operators $\Gamma_{g_0,g_i} : H^{g_i} \to H^{g_0}$ by block matrix decomposition

$$\Gamma_{g_0,g_k} = \left[\sqrt{\alpha_1}\Gamma_{p_1}|\sqrt{\alpha_2}\Gamma_{p_2}|\ldots|\sqrt{\alpha_k}\Gamma_{p_k}\right],$$
$$\Gamma_{g_0,g_{k+l}} = \left[\sqrt{\beta_1}\Gamma_{q_1}|\sqrt{\beta_2}\Gamma_{q_2}|\ldots|\sqrt{\beta_l}\Gamma_{q_l}\right],$$
$$\Gamma_{g_0,g_{k+l+m}} = \left[\sqrt{\delta_1}\Gamma_{s_1}|\sqrt{\delta_2}\Gamma_{s_2}|\ldots|\sqrt{\delta_m}\Gamma_{s_m}\right].$$

Denote by $\mathcal{O}_{0,H}$ the operator from zero subspace to H and by $\mathcal{O}_{H,0}$ the zero operator from H to zero subspace. Put

$$\begin{aligned}\Gamma_{g_{k-1},g_k} &= \sqrt{\alpha_k - \alpha_1}I_{H_{p_1}} \oplus \sqrt{\alpha_k - \alpha_2}I_{H_{p_2}} \oplus \ldots \oplus \\ &\quad \sqrt{\alpha_k - \alpha_{k-1}}I_{H_{p_{k-1}}} \oplus \mathcal{O}_{H_{p_k},0},\\ \Gamma_{g_{k-1},g_{k-2}} &= \mathcal{O}_{0,H_{p_1}} \oplus \sqrt{\alpha_2 - \alpha_1}I_{H_{p_2}} \oplus \ldots \oplus \sqrt{\alpha_{k-1} - \alpha_1}I_{H_{p_{k-1}}},\\ \Gamma_{g_{k-3},g_{k-2}} &= \sqrt{\alpha_{k-1} - \alpha_2}I_{H_{p_2}} \oplus \sqrt{\alpha_{k-1} - \alpha_3}I_{H_{p_3}} \oplus \ldots \oplus \\ &\quad \sqrt{\alpha_{k-1} - \alpha_{k-2}}I_{H_{p_{k-2}}} \oplus \mathcal{O}_{H_{p_{k-1}},0},\\ &\ldots.\end{aligned} \tag{2}$$

Operators for the rest two branches of the graph G can be built analogously. One can verify that thus constructed operators $\Gamma_{g_ig_j} : H^{g_j} \to H^{g_i}$ define a locally scalar representation Π of the graph G ($\Pi(\gamma_{g_i,g_j}) = \{\Gamma_{g_i,g_j}, \Gamma_{g_j,g_i}\}$) with the following character f: $f(g_0) = \gamma$

$$
\begin{array}{lll}
f(g_k) = \alpha_k, & f(g_{k+l}) = \beta_l, & f(g_{k+l+m}) = \delta_l, \\
f(g_{k-1}) = \alpha_k - \alpha_1, & f(g_{k+l-1}) = \beta_l - \beta_1, & f(g_{k+l+m-1}) = \delta_m - \delta_1, \\
f(g_{k-2}) = \alpha_{k-1} - \alpha_1, & f(g_{k+l-2}) = \beta_{l-1} - \beta_1, & f(g_{k+l+m-2}) = \delta_{m-1} - \delta_1, \\
f(g_{k-3}) = \alpha_{k-1} - \alpha_2, & f(g_{k+l-3}) = \beta_{l-1} - \beta_2, & f(g_{k+l+m-3}) = \delta_{m-1} - \delta_2, \\
f(g_{k-4}) = \alpha_{k-2} - \alpha_2, & f(g_{k+l-4}) = \beta_{l-2} - \beta_2, & f(g_{k+l+m-4}) = \delta_{m-2} - \delta_2, \ldots \\
\ldots & \ldots &
\end{array}
\qquad (3)
$$

Viceversa, if Π is a locally scalar representation of the graph G with character $f(g_i) = x_i$ which corresponds to non-degenerate representation π of algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ then it is routine to check that

$$
\begin{aligned}
\alpha_k &= x_k, \\
\alpha_1 &= x_k - x_{k-1}, \\
\alpha_{k-1} &= x_k - x_{k-1} + x_{k-2}, \\
\alpha_2 &= x_k - x_{k-1} + x_{k-2} - x_{k-3}, \\
\alpha_{k-3} &= x_k - x_{k-1} + x_{k-2} - x_{k-3} + x_{k-4}, \\
&\ldots .
\end{aligned}
\qquad (4)
$$

Analogously we can find β_j and δ_j. Further on we will denote Π as $\Phi(\pi)$.

Let π and $\widetilde{\pi}$ be non-degenerate *- representations of the algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ and C_0 - intertwining operator, i.e. $C_0\pi = \pi' C_0$. Put

$$
\begin{aligned}
C_{p_i} &= \widetilde{\Gamma}^*_{p_i} C_0 \Gamma_{p_i}, C_{p_i} : H_{p_i} \to \widetilde{H}_{p_i}, \\
C_{q_i} &= \widetilde{\Gamma}^*_{q_i} C_0 \Gamma_{q_i}, C_{q_i} : H_{q_i} \to \widetilde{H}_{q_i}, \\
C_{s_i} &= \widetilde{\Gamma}^*_{s_i} C_0 \Gamma_{s_i}, C_{s_i} : H_{s_i} \to \widetilde{H}_{s_i}. \\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
C^{g_0} &= C_0 : H^{g_0} \to \widetilde{H}^{g_0}, \\
C^{g_k} &= C_{p_1} \oplus C_{p_2} \oplus \ldots \oplus C_{p_k} : H^{g_k} \to \widetilde{H}^{g_k}, \\
C^{g_{k-1}} &= C_{p_1} \oplus C_{p_2} \oplus \ldots \oplus C_{p_{k-1}} : H^{g_{k-1}} \to \widetilde{H}^{g_{k-1}}, \\
C^{g_{k-2}} &= C_{p_2} \oplus C_{p_3} \oplus \ldots \oplus C_{p_{k-1}} : H^{g_{k-2}} \to \widetilde{H}^{g_{k-2}}, \\
C^{g_{k-3}} &= C_{p_2} \oplus C_{p_3} \oplus \ldots \oplus C_{p_{k-2}} : H^{g_{k-3}} \to \widetilde{H}^{g_{k-3}}, \\
&\ldots
\end{aligned}
$$

Analogously we can build operators C^{g_i} for $i \in \{k+1, \ldots, k+l+m\}$. It can be verified that family of operators $\{C^{g_i}\}_{0 \le i \le k+l+m}$ intertwines representations $\Pi = \Phi(\pi)$ and $\widetilde{\Pi} = \Phi(\widetilde{\pi})$ (morphism in category $\mathrm{Rep}(G)$). Put $\Phi(C_0) = \{C^{g_i}\}_{0 \le i \le k+l+m}$. One can check that Φ respect compositions of morphisms and identity operators and thus is a functor $\Phi : \overline{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma} \to \mathrm{Rep}(G)$. Its not hard to verify also that functor Φ is full and univalent.

Let Π be finite dimensional locally-scalar representation of the graph G with character f and dimension d ($f(g_i) = x_i, d(g_i) = d_i$).

It is almost obvious that representation Π is isomorphic (unitary equivalent) to a representation in the image of Φ if and only if

$$\begin{aligned} &1.\ 0 < x_1 < x_2 < \ldots < x_k; 0 < x_{k+1} < x_{k+2} < \ldots < x_{k+l};\\ &\qquad 0 < x_{k+l+1} < x_{k+l+2} < \ldots < x_{k+l+m};\\ &2.\ 0 < d_1 < d_2 < \ldots < d_k < d_0; 0 < d_{k+1} < d_{k+2} < \ldots <\\ &\qquad d_{k+l} < d_0; 0 < d_{k+l+1} < d_{k+l+2} < \ldots < d_{k+l+m} < d_0. \end{aligned} \tag{5}$$

(All the matrices of representation of G except matrices $\Gamma_{g_0,g_k}, \Gamma_{g_k,g_0}, \Gamma_{g_0,g_{k+l}}, \Gamma_{g_{k+l},g_0}$, $\Gamma_{g_0,g_{k+l+m}}, \Gamma_{g_{k+l+m},g_0}$ can be reduced by admissible transformations to the canonical form (2), then the rest of matrices naturally splits into blocks which will define matrices $\Gamma_{p_i}, \Gamma_{q_i}, \Gamma_{s_i}$, and thus projectors P_i, Q_i, R_i). A representation Π which satisfies condition (5) we will call non-degenerate representation of graph G. The full subcategory in $\operatorname{Rep} G$ of such representations we will denote by $\overline{\operatorname{Rep}}G$ and the full subcategory of irreducible non-degenerate representations by $\widetilde{\operatorname{Rep}}G$

Let

$$\begin{aligned} \dim H_{p_i} &= n_i, 1 \le i \le k;\\ \dim H_{q_i} &= n_{k+i}, 1 \le i \le l;\\ \dim H_{s_i} &= n_{k+l+i}, 1 \le i \le m;\\ \dim H_0 &= n_0. \end{aligned}$$

$(n_0, n_1, \ldots, n_{k+l+m})$ - generalized dimension of the representation π of the algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$. Let $\Pi = \Phi(\pi)$ where π is non-degenerate and $(d_0, d_1, \ldots, d_{k+l+m})$ dimension of Π. It is easy to see that

$$\begin{aligned} n_1 + n_2 + \ldots + n_k &= d_k,\\ n_1 + n_2 + \ldots + n_{k-1} &= d_{k-1},\\ n_2 + \ldots + n_{k-1} &= d_{k-2},\\ n_2 + \ldots + n_{k-2} &= d_{k-3},\\ &\ldots \end{aligned}$$

Thus

$$\begin{aligned} n_k &= d_k - d_{k-1},\\ n_1 &= d_{k-1} - d_{k-2},\\ n_{k-1} &= d_{k-2} - d_{k-3},\\ n_2 &= d_{k-3} - d_{k-4},\\ &\ldots \end{aligned} \tag{6}$$

Analogously we can find $n_{k+1}, \ldots, n_{k+l}$ if $d_{k+1}, \ldots, d_{k+l}$ are given and $n_{k+l+1}, \ldots, n_{k+l+m}$ given $d_{k+l+1}, \ldots, d_{k+l+m}$. Denote by $\Phi(\alpha, \beta, \delta, \gamma)$ the character defined by formulae (3). Let $\widetilde{\operatorname{Rep}}(G, f)$ denotes the category of irreducible non-degenerate locally-scalar representations of the graph G with fixed character f. Since all irreducible representations of a Dynkin graph are finite dimensional the restriction of the functor Φ on the category $\widetilde{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ defines an equivalence of this category with $\widetilde{\mathrm{Rep}}(G, f_\Phi)$ where $f_\Phi = \Phi(\alpha, \beta, \delta, \gamma)$. Thus we have proven the following

Theorem 1. *Let $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ be a *-algebra associated with graph G. Functor Φ is a full and unipotent functor from the category $\overline{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ of non-degenerate *-representations to the category $\mathrm{Rep}\, G$ of locally -scalar representations of graph G. If G is a Dynkin graph then the restriction of Φ on the full subcategory $\widetilde{\mathrm{Rep}}\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ of irreducible non-degenerate *-representations defines an equivalence with the category $\widetilde{\mathrm{Rep}}(G, f_\Phi)$ of irreducible non-degenerate locally-scalar representations of the graph G with fixed character $f_\Phi = \Phi(\alpha, \beta, \delta, \gamma)$*

The essential role for classification of locally scalar graph representations is plaid by Coxeter functors F° and $F^\bullet$ of odd and even reflections (see [5]). It was shown that all irreducible locally-scalar representations of Dynkin diagram could be obtained by repeated application of these functors starting from simple ones corresponding to simple roots. We can define now Coxeter functor for *-algebras $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ putting $\Psi^\circ = \Phi^{-1}F^\circ\Phi$ and $\Psi^\bullet = \Phi^{-1}F^\bullet\Phi$. They can be used to obtain irreducible representations of the algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ (associated with Dynkin graph) starting from the "simplest"ones.

3. Algebra associated with Dynkin graph $\mathcal{D}_4$

Let $\lambda = (\alpha, \beta, \delta, \gamma)$ be a vector of parameters of algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ associated with Dynkin graph $\mathcal{D}_4$ where $\alpha, \beta, \delta, \gamma \in \mathbb{R}^*$. Let ν be a permutation on the set $\{\alpha, \beta, \delta\}$. It is obvious that *-algebras $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ and $\mathcal{P}_{\nu(\alpha),\nu(\beta),\nu(\delta),\gamma}$ are isomorphic. Thus we will assume that $\alpha \le \beta \le \delta$. Denote by D_λ the set of generalized dimensions (d_0, d_1, d_2, d_3) of non-degenerated *-representations of the algebra $\mathcal{P}_\lambda$, $D(\mathcal{D}_4) = \cup_{\lambda \in W(\mathcal{D}_4)} D_\lambda$. Let $\mathbb{N}_\infty$ denote the set $\mathbb{N} \cup \{\infty\}$.

The following theorem gives complete description of $W(\mathcal{D}_4)$ and the sets of generalized dimensions D_λ for each $\lambda \in W(\mathcal{D}_4)$.

Theorem 2. *For algebra $\mathcal{P}_\lambda$ associated with Dynkin graph $\mathcal{D}_4$($\lambda = (\alpha, \beta, \delta, \gamma) \in (\mathbb{R}^+)^4$, $0 < \alpha \le \beta \le \delta$) the set $W(\mathcal{D}_4)$ can be decomposed into union of non-intersecting subsets $W(\mathcal{D}_4) = W_1 \cup W_2 \cup W_3 \cup W_4$ and the set $D(\mathcal{D}_4)$ of dimensions (d_0, d_1, d_2, d_3) of representations will be decomposed correspondingly $D(\mathcal{D}_4) = D_1 \cup D_2 \cup D_3 \cup D_4$ (if $\lambda \in W_i$ then $D_\lambda = D_i$) where*

(1) $W_1 = \{\lambda = (\alpha, \beta, \delta, \gamma) \in (\mathbb{R}^+)^4 | 0 < \alpha \le \beta \le \delta < \gamma, \alpha + \beta + \delta = 2\gamma\}$
$D_1 == \{(2m, m, m, m) \in (\mathbb{N}_\infty)^4\}$.
(2) $W_2 = \{\lambda = (\alpha, \alpha, \alpha, \alpha) \in (\mathbb{R}^+)^4\}$
$D_2 == \{(d_1 + d_2 + d_3, d_1, d_2, d_3) \in (\mathbb{N}_\infty)^4\}$.
(3) $W_3 = \{\lambda = (\alpha, \alpha, \alpha, 2\alpha) \in (\mathbb{R}^+)^4\}$
$D_3 = \{(d_1 + d_2 + d_3, d_1 + d_2, d_1, d_2 + d_3) \in (\mathbb{N}_\infty)^4\}$.
(4) $W_4 = \{\lambda = (\alpha, \beta, \alpha + \beta, \alpha + \beta) \in (\mathbb{R}^+)^4 | 0 < \alpha \le \beta\}$
$D_4 = \{(d_1 + d_2, d_1, d_1, d_2) \in (\mathbb{N}_\infty)^4\}$.

Proof As follows from the results obtained in the first section and in paper [5] all non-degenerate irreducible representations of the algebra associated with graph $\mathcal{D}_4$ are two dimensional and exist only for parameters in domain W_1. Degenerate representation must be one dimensional since the image of such representation is generated by two projections connected by non trivial linear relation. It is obvious that if the algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ has one dimensional representations then parameters are connected by on of the following set of relations: $\alpha = 0, \beta + \delta = \gamma$ or $\alpha = \beta = 0, \delta = \gamma$ or $\alpha + \beta + \delta = \gamma$. It is easy to see that none of them is compatible with relations on W_1. Thus for each algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ either all irreducible representations are multiple of unique two dimensional representation or are one dimensional. The letter case leads to subsets W_2, W_3, W_4. □

REFERENCES

[1] W.Fulton, *Eigenvalues, invariant factors, highest weights and Schubert calculus* // Bull. Amer. Math. Soc. **37** (2000), 209-249

[2] A. A. Klyachko, *Stable bundles, representation theory and Hermitian operators* // Sel. Math. , New ser. 4 (1998) 419 - 445

[3] С.В. Попович, Ю.С. Самойленко, *О гомоморфизмах алгебр, порожденных проекторами, и функторах Коксmepa*, Укр. Мат. Жур. -т.55, 9, 2003, с. 1224-1237

[4] Власенко М., Мелит А., Самойленко Ю., *Об алгебрах, порожденных линейно связанными образующими с заданым спектром* подано в Функ. Анал. и прилож.

[5] С.А. Кругляк, И.К. Редчук, А.В. Ройтер, *Локально-скалярные представления и разделяющие функции* / Киев 2003.-62с(Препр. / НАН украинi. Ин-т. математики; 2003.8)

[6] W. Crawley-Boevey and Ch. Geiss, *Horn's problem and semi-stability for quiver representations* // Representations of Algebars.-V.I. Poc. IX Intern. Conf. Benjing21 aug.-1 sept.2000 // Red. T.Happel, V.B. Zhong Benjing, 2002, p.40-48

[7] W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero* // Duke Math. J. -**118**, - N 2 (2003), p. 339-352