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RISK IN NON-COOPERATIVE GAME UNDER UNCERTAINTY

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The peculiarity of taking into account of risk in non-cooperative game under uncertainty is revealed. We assume that the limits of changes are the only thing known about these uncertainties.

1. Statement of the problem.

Consider a non-cooperative game of N players under uncertainty

$$< \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, Y, \{f_i(x,y)\}_{i\in\mathbf{N}} > . \qquad (1)$$

Here $\mathbf{N} = \{1,..,N\}$ is the set of *players numbers*. Every i-th player ($i \in \mathbf{N}$) chooses his *strategy* $x_i \in X_i \subset \mathbf{R}^{n_i}$. As a result the *game situation* $x = (x_1,..,x_N) \in X = \prod_{i\in\mathbf{N}} X_i$ is composed. Also some *uncertainty* $y \in Y \subset \mathbf{R}^m$ is realized independently of the players choice. On the set $X \times Y$ the *payoff function* $f_i(x,y)$ $i \in \mathbf{N}$ of the i-th player is determined. The value of this function on the concrete couple (x,y) is said to be the player's payoff.

The player's objective point in the game is to choose his strategy so that his payoff will be as large as possible. When choosing their strategy the players should allow for emerging of any uncertainty $y \in Y$ unpredictable in advance.

For the game the optimal solution based on the suitable modification of Wald principle has already been proposed (see [1]) (principle of maximin utility). However the usage of this approach orients the players to the "catastrophe". As a rule the probability of catastrophe appearance is small. Therefore for the game (1) a new concept of guaranteed solution is offered. This concept is based on the suitable modification of minimax regret principle (see [2]).

The following designations are used:

X_i^Y is the set of functions $x_i(y)$ determined on Y with values from X_i; situations $(x) = (x_{\mathbf{N}\setminus i}, x_i)$; N-vector-columns $\Phi^{(r)} = (\Phi_1^{(r)},..,\Phi_N^{(r)})$ $(r = 1,2)$; $(\Phi^{(1)} < \Phi^{(2)}) \Leftrightarrow (\Phi_i^{(1)} < \Phi_i^{(2)}, i \in \mathbf{N})$; $(\Phi^{(1)} \not< \Phi^{(2)}) \Leftrightarrow \rceil(\Phi^{(1)} < \Phi^{(2)})$; $X_{\mathbf{N}\setminus i} = \prod_{j\in\mathbf{N}\setminus i} X_j$. Further every payoff function $f_i(x,y)$ is put into correspondence to risk-function of the i-th player

$$\Phi_i(x,y) = \max_{z_i\in X_i} f_i(x_{\mathbf{N}\setminus i}, z_i, y) - f_i(x,y) \quad (i \in \mathbf{N}). \qquad (2)$$

This function shows the regret of the i-th player. The regret means the following. For the couple $(x,y) \in X \times Y$ composed in the game the player i has used the strategy x_i but not *arg* $\max_{z_i\in X_i} f_i(x_{\mathbf{N}\setminus i}, z_i, y)$. Every player aims to get the smallest possible value of his risk-function. Besides, the players should allow for emerging of any uncertainty $y \in Y$.

Let the game (1) the non-cooperative N-person game under uncertainty correspond to

$$< \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, Y, \{\Phi_i(x,y)\}_{i\in\mathbf{N}} > . \qquad (3)$$

Here $\mathbf{N}, X_i, Y$ are the same as in (1). The payoff function $\Phi_i(x, y)$ of the i-th player is of the form (2) and coincides with player's risk-function. Every i-th player chooses $x_i \in X_i$ so that his risk-function $\Phi_i(x, y)$ value will be as less as possible.

The following definition is based on the combination of the Nash equilibrium notion (from the theory of non-cooperative games (see [3])) and the notion of maximum by Slater (from the theory of multicriteria problems (see[4])).

Definition. The couple $(x^e, \Phi^*) \in X \times \mathbf{R}^N$ is said to be the *guaranteed R-solution* of the game (1) if there exists an uncertainty $y^* \in Y$ such that $\Phi^* = \Phi(x^e, y^*)$ and

$$\Phi_i(x^e_{\mathbf{N}\setminus i}, x_i, y^*) \geq \Phi_i(x^e, y^*), \quad \forall x_i \in X_i \quad (i \in \mathbf{N}), \tag{4}$$

$$\Phi_i(x^e, y^*) \not< \Phi_i(x^e, y), \quad \forall y \in Y. \tag{5}$$

The *situation* x^e is called the *R-guaranteed Nash equilibrium*. And the vector $\Phi(x^e, y^*)$ is called the *R-guaranteed risk in the game (1)*.

Remark 1. The situation x^e satisfying (4) is the Nash equilibrium in the non-cooperative game $< \mathbf{N}, \{X_i\}_{i \in \mathbf{N}}, \{\Phi_i(x, y^*)\}_{i \in N} >$. Hence by (2) and in view of the fact that $\max_{z_i} f_i(x_{\mathbf{N}\setminus i}, z_i, y)$ does not depend on x_i we get that x^e satisfies the conditions

$$f_i(x^e_{\mathbf{N}\setminus i}, x_i, y^*) \leq f_i(x^e, y^*), \quad \forall x_i \in X_i \quad (i \in \mathbf{N}). \tag{6}$$

This means that x^e is the Nash equilibrium in the game

$$< \mathbf{N}, \{X_i\}_{i \in \mathbf{N}}, \{f_i(x, y^*)\}_{i \in \mathbf{N}} > \tag{7}$$

obtained from (3) by fixing $y = y^*$.

Remark 2. The uncertainty y^* from (5) is maximal by Slater in the multicriteria problem $< Y, \{\Phi_i(x^e, y)\}_{i \in \mathbf{N}} >$ obtained from (3) by fixing $x = x^e$.

Remark 3. The game sense of the R-solution (x^*, Φ^*) is the following. Using the strategies x^e_i $(i \in \mathbf{N})$ from the R-guaranteed Nash equilibrium the players provide themselves the R-guaranteed risk $\Phi^* = (\Phi_1(x^e, y^*), ..., \Phi_N(x^e, y^*))$. Namely, whatever uncertainty $y \in Y$ is realized, the risks $\Phi_i(x^e, y)$ can not become bigger than the corresponding components of the risk Φ^* (i.e.$\Phi^* \not< \Phi(x^e, y)$).

2. Existence Theorem

Theorem. Let in the game (1)

1^0 X_i $(i \in \mathbf{N})$ be nonempty, convex and compact sets, and Y be a compact set,

2^0 payoff functions $f_i(x_{\mathbf{N}\setminus i}, x_i, y)$ be continuous on $X \times Y$ and be strictly concave in x_i for every fixed $(x_{\mathbf{N}\setminus i}, y) \in X_{\mathbf{N}\setminus i} \times Y$.

Then the guaranteed R-solution of the game (1) is a couple $(x^e, 0_N)$ where x^e is the Nash equilibrium situation in the game (7) (i.e. x^e satisfies (6)) and 0_N is a zero N-vector.

Доказательство. Let us consider a set of non-cooperative games

$$\Gamma(y) =< \mathbf{N}, \{X_i\}_{i \in \mathbf{N}}, \{f_i(x, y)\}_{i \in \mathbf{N}} > . \tag{8}$$

corresponding to various uncertainties $y \in Y$. According to [3, p.90], for each $y \in Y$ there exists "its own"Nash equilibrium situation $x^e(y)$ satisfying the equality

$$\max_{z_i \in X_i} f_i(x^e_{\mathbf{N}\setminus i}(y), z_i, y) = f_i(x^e_{\mathbf{N}\setminus i}(y), x^e_i(y), y), \quad i \in \mathbf{N}. \tag{9}$$

Note, that strict concavity of $f_i(x,y)$ in x_i implies for each $y \in Y$ the strategy $x_i^e(y) \in X_i^Y$ determined by (9) is unique.

The compactness of the sets X and Y, continuity on $X \times Y$ and strict concavity in x_i of the function $f_i(x,y)$ imply the existence of only one realization $x_i(x_{\mathbf{N}\setminus i}, y)$ of maximum

$$\max_{z_i \in X_i} f_i(x_{\mathbf{N}\setminus i}, z_i, y) = f_i(x_{\mathbf{N}\setminus i}, x_i(x_{\mathbf{N}\setminus i}, y), y) \tag{10}$$

for each $(x_{\mathbf{N}\setminus i}, y) \in X_{\mathbf{N}\setminus i} \times Y$. The function $x_i(x_{\mathbf{N}\setminus i}, y)$ is continuous (see [5, p.54]). From (10) for $x_j = x_j^e(y) \in X_j^Y$ and $x_j^e(y)$, satisfying (9), we get

$$\max_{z_i \in X_i} f_i(x^e_{\mathbf{N}\setminus i}(y), z_i, y) = f_i(x^e_{\mathbf{N}\setminus i}(y), x_i(x^e_{\mathbf{N}\setminus i}(y), y), y) \tag{11}$$

for each $y \in Y$. Since the left parts of (9) and (10) are equal the right parts are equal to:

$$f_i(x^e_{\mathbf{N}\setminus i}(y), x_i^e(y), y) = f_i(x^e_{\mathbf{N}\setminus i}(y), x_i(x^e_{\mathbf{N}\setminus i}(y), y), y) \quad \forall y \in Y. \tag{12}$$

In view of the risk-functions (2) from (10)-(12) we get

$$\Phi_i(x^e(y), y) = 0, \quad \forall y \in Y \quad (i \in \mathbf{N}).$$

□

Remark 4. From the theorem we obtain an important game fact: if for each uncertainty $y \in Y$ the players find the Nash equilibrium situation $x^e(y) \in X$ in the game (8) and use it, they provide themselves a zero-risk when any of the uncertainties $y \in Y$ is realized in the game (1).

Remark 5. From the theorem we get the following scheme of the guaranteed solution (x^e, Φ^*) construction for the game (1):

– to construct a mathematical model of the investigated problem in the form of the ordered set (1);

– by (9) for each $y \in Y$ to construct the Nash equilibrium situation $x^e(y)$.

Then for each $y \in Y$ the couple $(x^e(y), 0_N) \in X \times \mathbf{R}^N$ is a guaranteed R-solution of the game (1). Namely, whatever uncertainty $y^* \in Y$ is realized the players applying the strategies $x_i^e(y^*)$ $(i \in \mathbf{N})$ from the Nash equilibrium situation $x^e(y^*)$ ($x^e(y^*)$ satisfies (9)), provide themselves a zero-risk (the best risk from all possible in the game (1)).

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