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ON SOLVABILITY AND EXTREME REGULARIZATION OF VARIATIONAL INEQUALITIES WITH SET-VALUED OPERATORS

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In this paper we modify a solvability theorem for variational inequalities with multivalued operators. We propose also to solve such problems using some special inclusion or a sequence of some extreme problems, so-called "extreme regularization"of problem. These extreme problems consist of inclusions with a parameter and a special penalty function. The extreme regularization method is applied for variational inequalities with set-valued mappings, which has more weak boundedness conditions with respect to ones in previous papers. Results are also applied for variational inequalities with single-valued mappings.

Ключевые слова: variational inequality, inclusion, extreme regularization, penalty function, property (M), quasi-boundedness, local boundedness

1. The variational inequalities theory is powerful and effective tool for studying a wide class of free boundary problems, equilibrium problems, nonlinear optimization problems, etc. These inequalities have important applications to various branches of pure mathematics and applied sciences, in particular, to economics and engineering (see e.g. [3]). Wide class of models consist of essentially set-valued variational inequalities, for example, a family of control problems which are described by variational inequalities, minimax problems, nondifferential optimal problems etc. (see e.g. [1]). The immediate solving of such problems is very difficult. In [5, 7, 8] it was proposed the idea of extreme regularization for variational inequalities that allows to replace a variational inequality by auxiliary extreme problem. In [10] it was shown that for solvability of inclusions and variational inequalities it is sufficient to use weaker boundedness and coercivity conditions than in [2, 6]. In this paper we prove that in that case solution's sets of auxiliary extreme problems coincide with solution's sets of corresponding variational inequality.

2. Let X be a reflexive Banach space, let X^* be its topological dual space, and let $\langle\cdot,\cdot\rangle$ be the duality pairing on $X \times X^*$. Denote by $Conv(X^*)$ the totality of all nonempty convex closed subsets of the space X^*. Let A be a multivalued mapping such that

$$\mathrm{Dom}(A) = \{y \in X : A(y) \neq \emptyset\} = X.$$

Denote $\mathrm{graph}(A) = \{(y, w) \in X \times X^* : w \in A(y)\}$. We introduce upper and lower support functions, and an upper norm associated with A by the formulas

$$[A(y), \xi]_+ = \sup_{d \in A(y)} \langle d, \xi \rangle, \quad [A(y), \xi]_- = \inf_{d \in A(y)} \langle d, \xi \rangle, \quad \|A(y)\|_+ = \sup_{d \in A(y)} \|d\|_{X^*}.$$

We consider the following variational inequality with set-valued mapping

$$[A(y), \xi - y]_+ \;\geq\; \langle f, \xi - y \rangle \qquad \forall \xi \in K, \tag{1}$$

where $f \in X^*$, $K \subset X$ is a closed convex set, $\dim K = \dim X$.

Since the support functions define the values of operator within a convex closure [4], we use mappings with convex, closed values. Moreover, if variational inequality (1) is solvable and $A(y)$ is bounded, then there exists an element $w \in A(y)$ such that

$$\langle w, \xi - y\rangle \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K$$

(see [6]). Thus we suppose that the values of A are also bounded.

Definition 1. A mapping $A : X \to Conv(X^*)$ is said to be **generalized pseudomonotone** if for arbitrary $(y_n, w_n) \in \text{graph}(A)$ such that $y_n \to y$ weakly in X, $w_n \to w$ weakly in X^*, and $\limsup\limits_{n\to\infty}\langle w_n, y_n - y\rangle \leq 0$ we have $w \in A(y)$ and $\langle w_n, y_n\rangle \to \langle w, y\rangle$ (to within a subsequence).

Definition 2. A mapping $A : X \to Conv(X^*)$ is said to be **quasi–bounded** if for all

$$\{(y, w) \in \text{graph}(A) : \|y\|_X \leq k_1, \quad \text{and} \quad \langle w, y - \zeta\rangle \leq k_2 \qquad \text{for some } k_1 > 0,\ k_2 > 0,\ \zeta \in X\}$$

there exists a constant $N > 0$ such that $\|w\|_{X^*} \leq N = N(k_1, k_2, \zeta) < \infty$.

We also use properties of the mapping A on finite dimensional subspaces. Let $F \subset X$ be an arbitrary finite–dimensional space with a basis $\{h_i\}$ and with a topology, which is induced by topology of X. We introduce a projecting operator $I_F : F \to X$ ($\|I_F y_F\|_X = \|y_F\|_F \,\forall y_F \in F$), and the adjoint operator $I_F^* : X^* \to F^*$. We denote

$$f_F = \sum_{\{h_i\}}\langle f, h_i\rangle h_i, \qquad A_F \equiv A|_F : F \to Conv(X^*),$$

$$I_F^* A_F(y) = \bigcup_{d \in A_F(y)} \left\{\sum_{\{h_i\}}\langle d, h_i\rangle h_i\right\} \qquad \forall y \in F.$$

Definition 3. A mapping $G : F \to Conv(F)$ is **locally bounded** if for any $y \in F$ there exist $\varepsilon > 0$ and $M > 0$ such that $\sup\limits_{d\in G(\xi)} \|d\|_F \leq M$ for any $\xi \in \{\xi \in \text{Dom}(G) : \|\xi - y\|_F \leq \varepsilon\}$.

Definition 4. A mapping $G : F \to Conv(F)$ is **upper semicontinuos** if for every $\varepsilon > 0$ and $y \in \text{Dom}(G)$ there exists $\delta > 0$ such that $A(z) \subset A(y) + B_\varepsilon(0)$ for any $z \in B_\delta(y)$, where $B_\varepsilon(0)$ is a ball of center 0 and radius ε.

We also consider some convex bounded set $D \subset X$ such that $\dim D = \dim X$. Let also ζ_0 be an interior point of D with respect to topology of F, i.e. $\zeta_0 \in \text{int}_F(F \cap D) \equiv \text{int}_F D_F$.

3. First we consider a solvability of variational inequalities.

Theorem 1. *Let X be a reflexive Banach space; let $A : K \to Conv(X^*)$ be a quasi–bounded, generalized pseudomonotone operator such that for any finite-dimensional space F a mapping $I_F^* A_F : F \to Conv(F)$ is locally bounded or upper semicontinuous; let $K \subset X$ be a closed, convex set ($\dim K = \dim X$). Moreover, one of the following conditions holds:*

i) (an acute angle's condition) there exist some bounded, convex set D and an element $\zeta_0 \in D \cap K$ such that for any finite–dimensional $F \ni \zeta_0$ we have $\zeta_0 \in \text{int}_F(D \cap K)_F$ and

$$[I_F^* A_F(y) - f_F, y_F - \zeta_0]_+ \geq 0 \qquad \forall y \in \partial D_F, \tag{2}$$

where ∂D_F is the boundary of $D_F = D \cap F$ in space F;

ii) K is bounded.

Then a set of solutions for variational inequality

$$[A(y), \xi - y]_+ \ \geq\ \langle f, \xi - y\rangle \qquad \forall \xi \in K \cap D \tag{3}$$

is nonempty and weakly compact in $D \cap K$. Moreover, each solution of (3) is a solution of (1).

Доказательство. We can prove this theorem similarly to Theorem 2 [10] for bounded mappings.

As it was proved in Lemma 2 [10], a solution's set of variational inequality (3) is equal to a solution's set of the following inclusion

$$A(y) + N_K^\lambda(y) \ni f, \qquad \lambda = N(k_1, 0, \zeta_0), \tag{4}$$

where $N : \Re_+ \times \Re_+ \times X \to \Re_+$ is defined in Definition 2, $B_\lambda(0) = \{w \in X^* : \|w\|_{X^*} \le \lambda\}$, and

$$N_K^\lambda(y) := \left\{ w \in B_\lambda(0) \subset X^* : \langle w, \xi \rangle \le 0 \qquad \forall \xi \in \bigcup_{h>0} \frac{1}{h}(y - K) \right\}$$

is a frustum of normal cone. Thus it is sufficient to solve inclusion (4).

The mapping $A + N_K^\lambda$ is quasi-bounded, and $A + N_K^\lambda$ has property (M), since it is a generalized pseudomonotone mapping (see Proposition 2 [9] or §3 [10]). Moreover, $I_F^*(A + N_K^\lambda)_F$ is a sum of upper semicontinuous mappings. Therefore this operator is upper semicontinuous. Hence it is sufficiently to show that condition (2) holds for $A + \lambda N_K^1$ on some bounded convex set $D \cap K$.

By definition, $0 \in N_K^\lambda(\xi)$ for $\xi \in K$, $\{0\} = N_K^\lambda(\xi)$ for $\xi \in \mathrm{int}K$, and $[N_K^\lambda(y), y - \zeta_0]_+ = \lambda\|y - \zeta_0\|_X > 0$ for $y \in \partial K$. If $y_F \in \partial(K \cap F) \subset F$, then $y = I_F y_F \in \partial K \subset X$. Moreover,

$$[A(y) - f + \lambda N_K^1(y), y - \zeta_0]_+ = [A(y) - f, y - \zeta_0]_+ + \lambda[N_K^1(y), y - \zeta_0]_+. \tag{5}$$

Let we consider the subsets ∂K_{DF} and ∂D_{FK} such that

$$\partial(K \cap D \cap F) = \partial K_{DF} \cup \partial D_{FK},$$
$$\partial K_{DF} = \{y_F \in \partial(K \cap F \cap D) : [I_F^* A_F(y_F) - f_F, y_F - \zeta_0]_+ < 0\},$$
$$\partial D_{FK} = \partial(K \cap D \cap F) \setminus \partial(K \cap F).$$

Obviously, by construction and by condition (2),

$$[I_F^* A_F(y_F) - f_F, y_F - \zeta_0]_+ \ge 0 \qquad \forall y_F \in \partial D_{FK}.$$

Moreover, by quasi-boundedness of A, we have estimate $\|A(y_F)\|_+ \le N$ for $y_F \in \partial K_{DF}$. Here N is independent of F, since $N = N(k_1, k_2, \zeta_0)$, $k_1 \le \sup_{\xi \in D \cap K} \|\xi\|_X$, $k_2 = 0$.

If we substitute $\lambda = N + \|f\|_{X^*}$ in (5), we obtain for any $y \in \partial K_{DF}$

$$[I_F^* A_F(y_F) - f_F + I_F^*(N_K^\lambda)_F(y_F), y_F - \zeta_0]_+ \ge (-N - \|f\|_{X^*} + \lambda)\|y - \zeta_0\|_X = 0.$$

Therefore, for arbitrary F

$$[I_F^* A_F(y_F) - f_F + I_F^*(N_K^\lambda)_F(y_F), y_F - \zeta_0]_+ \ge 0 \quad \text{for } y \in \partial(K \cap D \cap F).$$

By Theorem 2 [10], there exists $\hat{y} \in K \cap D$ such that $f \in A(\hat{y}) + N_K^\lambda(\hat{y})$.

If K is bounded, we can set $K \cap D = K$.

It remains to prove that the solution set is weakly compact in K ($K \cap D$). Let $y_n \to y$ weakly in X, and let y_n satisfy (3). Since $\limsup_{n\to\infty}\langle f, y_n - y \rangle \le 0$, using property (M), we have $f \in A(y) + N_K^\lambda(y)$. Thus the set of solutions is weakly compact. □

Definition 5. A mapping $A : X \to Conv(X^*)$ is " + " – **coercive (coercive)** if

$$\|y\|_X^{-1}[A(y), y - \zeta_0]_{+(-)} \to \infty \qquad \text{as } \|y\|_X \to \infty.$$

Obviously, "the acute angle's condition"(2) is weaker than coercivity condition or " + " – coercivity condition (see e.g. [4, 6, 10]). If A is " + "–coercive, it satisfies the "acute angle's condition (2) on some ball $B_R(\zeta_0)$. If A is coercive, it is " + "–coercive and there exists some ball $B_{\overline{R}}(\zeta_0)$ such that inequality (1) has no solutions outside of this ball (see e.g. [2, 4, 6, 9, 10]).

4. In general, the mapping N_K^λ (or $A + N_K^\lambda$) can have rather complicated form. In this case we can use the extreme regularization method.

Definition 6. A function $\beta : X \to \mathbf{R} \cup \{\infty\}$ is **lower semicontinuous** if

$$\liminf_{y_n \to y} \beta(y_n) \ge \beta(y).$$

Let us construct a penalty function

$$F(y,v) = \sup_{\xi \in K\cap D\cup\{y\}} \langle v, y-\xi\rangle + \beta(y) \equiv [v, y - K\cap D \cup \{y\}]_+ + \beta(y),$$

where y is a solution of the following inclusion

$$A(y) \ni f + v, \tag{6}$$

$\beta(y) = I_{K\cap D}(y)$ is an indicator of the set $K \cap D$. We can use an arbitrary lower semicontinuous function $\beta \geq 0$ such that $\beta(y) > 0$ if $y \notin K\cap D$, and $\beta(y) = 0$ if $y \in K \cap D$.

Now we can consider the following extreme problem

$$F(v,y) = \sup_{\xi \in K\cap D\cup\{y\}} \langle v, y-\xi\rangle + \beta(y) \to \inf_{v\in B_\lambda(0), y\in\rho(v)}, \tag{7}$$

where ρ is the solving mapping for inclusion (6), i.e. any $y \in \rho(v)$ satisfies (6).

Lemma 1. *Let problems (3) and (7) have solutions. Then y is a solution of (3) iff there exists v such that (y, v) is a solution of (7).*

Доказательство. Let y be a solution of (3). Then $\langle v, \xi - y\rangle \geq 0$ for arbitrary $\xi \in K \cap D$ and $\beta(y) = 0$, i.e. $F(v, y) = 0$. By construction,

$$F(v,y) \geq [v, y - D\cap K \cup \{y\}]_+ \geq \langle v, y - y\rangle = 0,$$

i.e. F is nonnegative. Thus (y, v) is a solution of (7).

Let (y, v) be a solution of (7). The function F is lower bounded by 0. By conditions of lemma, variational inequality (3) has at least one solution. Thus there exists a pair $(\hat{y}, \hat{v})$ such that $F(\hat{y}, \hat{v}) = 0$, i.e. a lower limit of values for F is accesible. Since (y, v) is a solution of (7), we obtain that $F(y, v) = 0$, $[v, y - K\cap D \cup \{y\}]_+ = 0$, and $\beta(y) = 0$ (i.e. $y \in K \cap D$). Thus y is a solution of (3). □

The problem (3) is solvable. Let us show that the extreme problem (7) is solvable too.

Theorem 2. *Let $K \subset X$ be a closed convex set,* $\dim K = \dim X$. *And let $A : X \to Conv(X^*)$ be a generalized pseudomonotone and quasi-bounded mapping such that for any finite-dimensional space F a mapping $I_F^* A_F : F \to Conv(F)$ is locally bounded or upper semicontinuous. Moreover, let K be bounded, or let A be " + "-coercive. Then for all $f \in X^*$ problem (7) has at least one solution.*

Доказательство. Since A is a " + "–coercive mapping, then for any $f, v \in X^*$ there exists a convex, bounded set D_v such that

$$[I_F^* A_F(y) - f_F - v_F, y_F - \zeta_0]_+ \geq 0 \qquad \forall y \in \partial(D_v)_F.$$

Thus by Theorem 2 [9], for each $v \in X^*$ there exists a solution of inclusion (6). Moreover, since we consider $v \in B_\lambda(0)$, then it is sufficient to choose a convex, bounded set $\widetilde{D}$, when $I_F^* A_F$ satisfies the strong acute condition (8). Such set exists by " + "-coercivity of mapping A. Hence for any $v \in B_\lambda(0)$ there exists $y(v) \in \widetilde{D}$, which satisfies inclusion (6). Therefore we can consider the function F on bounded set $B_\lambda(0) \times \widetilde{D}$. This function is bounded from below ($F(v, y) \geq 0$), and a lower limit of values is accesible. Let us consider a minimizing sequence $\{(v_n, y_n)\}$: $F(v_n, y_n) \to 0$. We denote $\xi_n = \text{argsup} \sup_{\xi\in K\cap D\cup\{y\}} \langle v_n, y_n - \xi\rangle$. By construction, the sequences $\{v_n\}$, $\{y_n\}$, and $\{\xi_n\}$ are bounded. Thus on reflexive Banach spaces there exist weakly convergent subsequences. Let $(y_m, v_m) \to (y, v)$ weakly on $X \times X^*$, where

$$\limsup_{m\to\infty} \langle v_m, y_m - y\rangle \leq \limsup_{m\to\infty} \langle v_m, y_m - \xi_m\rangle \leq \lim_{m\to\infty} F(v_m, y_m) = 0.$$

Since A is a generalized pseudomonotone map, we obtain $\langle v_m, y_m\rangle \to \langle v, y\rangle$ and $v + f \in A(y)$. Let us assume that $y \notin K \cap D$. But $K \cap D$ is weakly closed, i.e. there exist a neighborhood $U_y \subset X \setminus (K \cap D)$ in weak topology X and a number n_y such that $\{y_m\}_{m \geq n_y} \subset U_y$. Thus $\liminf_{m\to\infty} \beta(y_m) \geq \beta(y) > 0$. Hence y_m can not belong to a minimizing sequence. We obtain the contradiction. Consequently, y belongs to $K \cap D$ and satisfies (3) and $F(v, y) = 0$. □

Corollary 1. *Let $K \subset X$ be a unbounded, closed, convex set,* $\dim K = \dim X$. *And let $A : X \to Conv(X^*)$ be a generalized pseudomonotone and quasi-bounded mapping such that for any finite-dimensional space F a mapping $I_F^* A_F : F \to Conv(F)$ is locally bounded or upper semicontinuous. Moreover, let A satisfy so-called "strong acute angle's condition"*

there exist some bounded, convex set $\widetilde{D}$ and an element $\zeta_0 \in \widetilde{D} \cap K$ such that for any finite-dimensional $F \ni \zeta_0$ we have that $\zeta_0 \in \mathrm{int}_F(\widetilde{D} \cap K)_F$ and

$$[I_F^* A_F(y) - f_F, y_F - \zeta_0]_+ \geq \lambda \|y_F - \zeta_0\|_F \qquad \forall y \in \partial \widetilde{D}_F, \tag{8}$$

where $\partial \widetilde{D}_F$ is the boundary of $\widetilde{D}_F = \widetilde{D} \cap F$ in space F.

Then problem (7) has at least one solution.

Remark 13. Let K be a closed convex cone with corner at y_0, and let $\dim K = \dim X$. Then variational inequality (1) defines the following complementary problem

$$\text{find a pair } (y, d), \text{ where } y \in K,\ d \in K^* \cap A(y) - f,$$
$$K^* = \{w \in X^* : \langle w, \zeta - y_0\rangle \geq 0 \qquad \forall \zeta \in K\}.$$

Thus we can modify the penalty function from (7)

$$F(v, y) = |\langle v, y - y_0\rangle| + \beta(y)$$

(see [5]). In this case the results of Section 4 hold.

Remark 14. Note that in a definition of F. Browder [2] it was assumed that generalized pseudomonotone operators have restrictions to any finite-dimensional subspace that are upper semicontinuous.

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