Ученые записки Таврического национального университета им. В. И. Вернадского

Серия "Математика. Механика. Информатика и Кибернетика" N 1 (2003) 58 – 60
УДК 517.98

ON ENTIRE FUNCTIONS OF CARTWRIGHT CLASS WITH A FINITE NUMBER OF SINGULARITIES.

K. K. Simonov
Donetsk National University
Donetsk, Ukraine

The Cartwright class consists of entire function of finite exponential type such that the integral $\int_{-\infty}^{+\infty} \frac{\ln^+|f(t)|}{1+t^2}dt$ converges. In the present paper, a generalization of the Cartwright class for functions with a finite number of singularities is given. M. G. Krein's criterion for a function f to be from the Cartwright class is extended to the case where the function f has a finite number of singularities. A necessary and sufficient conditions for a function f with a finite number of singularities to be an element of a generalized Nevanlinna matrix is given.

Ключевые слова: Entire functions, Cartwright class, Nevanlinna functions, Nevanlinna matrices

1. Functions of Cartwright class

Let f be a holomorphic function in a simply connected domain D. The function f is called a Nevanlinna function if the function $\ln|f(z)|$ admits a positive harmonic majorant (see [6, 2]). Denote by $\mathcal{N}(D)$ the set of all Nevanlinna function in the domain D.

An entire function f is said to belong to the Cartwright class $\mathcal{C}$ if

$$\int_{-\infty}^{+\infty} \frac{\ln^+|f(t)|}{1+t^2}dt < \infty \quad \text{and} \quad \overline{lim}_{z\to\infty} \frac{\ln|f(z)|}{|z|} < \infty$$

(see [6, 8]).

The following theorem is due to M. G. Krein.

Theorem 1 (see [6]). *An entire function f belongs to the class $\mathcal{C}$ if and only if f belongs to the classes $\mathcal{N}(\mathbb{C}_+)$ and $\mathcal{N}(\mathbb{C}_-)$.*

In this note we generalize this result as well as several other results from [6, 7] to the case of holomorphic functions with a finite number of singularities. This generalization is motivated by some new interpolation problems considered recently in [3, 5, 9] (see also a survey [4]).

Take N points $\{x_j\}_1^N$ on the real line and set

$$\mathbf{G} = \mathbb{C} \setminus \{x_j\}_1^N .$$

Denote by $\mathcal{H}(\mathbf{G})$ the set of all holomorphic functions in the domain $\mathbf{G}$.

Definition 1. We say that a function $f \in \mathcal{H}(\mathbf{G})$ belongs to the generalized Cartwright class $\mathcal{C}(\mathbf{G})$ if

$$\int_{-\infty}^{+\infty} \frac{\ln^+|f(t)|}{1+t^2}dt < \infty, \tag{1}$$

$$\overline{lim}_{z\to\infty}\frac{\ln|f(z)|}{|z|}<\infty,\qquad \overline{lim}_{z\to\infty}|z-x_j|\ln|f(z)|<\infty\quad (j=1,2,\dots,N). \tag{2}$$

The following theorem allows us to restate many propositions for the class $\mathcal{C}(\mathbf{G})$.

Theorem 2. *Suppose a function $f\in\mathcal{H}(\mathbf{G})$ belongs to the class $\mathcal{N}(\mathbb{C}_+)$. Then f admits the representation*

$$f(z)=f_0(z)\cdot f_1\left(\frac{1}{x_1-z}\right)\cdot\dots\cdot f_N\left(\frac{1}{x_N-z}\right),$$

where f_0, f_1, ..., f_N are entire functions of class $\mathcal{N}(\mathbb{C}_+)$.

In particular, this implies the following generalization of Theorem 1.

Theorem 3. *A function $f\in\mathcal{H}(\mathbf{G})$ belongs to the class $\mathcal{C}(\mathbf{G})$ if and only if f belongs to the classes $\mathcal{N}(\mathbb{C}_+)$ and $\mathcal{N}(\mathbb{C}_-)$.*

2. Functions of Krein class

A matrix $W(z)=\begin{pmatrix}A(z)&B(z)\\C(z)&D(z)\end{pmatrix}$ is called a Nevanlinna matrix if the matrix elements A,B,C,D are entire transcendental functions and the following conditions hold:

(A) $A(z)D(z)-B(z)C(z)\equiv 1$.

(B) For any $t\in\mathbb{R}$, the function $w(z;t)=\dfrac{A(z)t+B(z)}{C(z)t+D(z)}$ is a holomorphic function in $\mathbb{C}\setminus\mathbb{R}$ and satisfies the inequality $\dfrac{\operatorname{Im}w(z;t)}{\operatorname{Im}z}\geq 0$ for any $z\in\mathbb{C}\setminus\mathbb{R}$.

See [1] for references to Nevanlinna matrices and their applications in the classical moment problem.

An entire function f belongs to the Krein class $\mathcal{K}$ if f is real, its zeros are real, and f admits the absolute convergent expansion

$$\frac{1}{f(z)}=\frac{A_0}{z}+z\sum_{k=1}^{\infty}\frac{A_k}{(z-\alpha_k)\alpha_k}+B. \tag{3}$$

The elements of Nevanlinna matrices are described by the following theorem due to M. G. Krein.

Theorem 4 (see [7]). *The set of all elements of Nevanlinna matrices coincides with the class $\mathcal{K}$ and is contained in the class $\mathcal{C}$.*

We generalize the class $\mathcal{K}$ to the functions with a finite number of singularities.

Definition 2. We say that a matrix function

$$W(z)=\begin{pmatrix}A(z)&B(z)\\C(z)&D(z)\end{pmatrix}\quad (z\in\mathbf{G})$$

is a generalized Nevanlinna matrix if its elements are transcendental functions holomorphic in $\mathbf{G}$, and the conditions (A) and (B) hold.

Definition 3. We say that a function $f\in\mathcal{H}(\mathbf{G})$ belongs to the class $\mathcal{K}(\mathbf{G})$ if f is real, its zeros $\{\alpha_k\}_1^\infty$ are real, and f admits the absolute convergent expansion

$$\frac{1}{f(z)}=\sum_{k=1}^{\infty}A_k\frac{1+\alpha_k z}{z-\alpha_k}+B+Cz, \tag{4}$$

where

$$\alpha_k, A_k, B, C \in \mathbb{R} \quad (k = 1, 2, 3, \ldots), \qquad \sum_{k=1}^{\infty} |A_k| < \infty.$$

Theorem 5. *The set of all elements of generalized Nevanlinna matrices coincides with the class $\mathcal{K}(\mathbf{G})$ and is contained in the class $\mathcal{C}(\mathbf{G})$.*

Remark 2. The class $\mathcal{K}(\mathbf{G})$ is invariant under the transformations

$$\zeta = \frac{1 + x_j z}{z - x_j} \quad (j = 1, 2, \ldots, N).$$

Remark 3. It is possible to describe the class $\mathcal{K}(\mathbf{G})$ with the alternative expansion

$$\frac{1}{f(z)} = \sum_{\alpha_k \in Z'} A_k \frac{1}{z - \alpha_k} + z \sum_{\alpha_k \in Z_\infty} \frac{A_k}{(z - \alpha_k)\alpha_k} + B + Cz,$$

instead of the expansion (4). Here the zero set of the function f is divided into two parts Z' and Z_∞ such that Z' is bounded and Z_∞ does not have finite accumulation points.

Remark 4. In the expansion (4), $C \neq 0$ if and only if the function f is holomorphic at the infinite point and $f(\infty) = 0$.

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, GIFML, Moskow, 1961.

[2] G. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[3] E. Hendriksen and C. Nijhuis, *Laurent-Jacobi matrices and the strong Hamburger moment problem*, Acta Appl. Math. **61** (2000), 119–132.

[4] William B. Jones and Olaf Njåstad, *Orthogonal Laurent polynomials and strong moment theory: a survey*, Journal of computational and applied mathematics **105** (1999), 51–91.

[5] I. S. Kats and A. A. Nudelman, *Strong Stieltjes moment problem*, St. Peterburg Math. J. **8** (1997), no. 6, 931–950.

[6] M. G. Krein, *On the theory of entire function of exponential type*, Izv. Akad. Nauk SSSR **11** (1947).

[7] M. G. Krein, *On the indefinite case of the Sturm–Liouville boundary problem in the interval* $(0, \infty)$, Izv. Akad. Nauk SSSR **16** (1952), no. 5, 293–324.

[8] B. Ya. Levin, *Lectures on entire functions*, American Mathematical Society, Providence, Rhode Island, 1996.

[9] K. K. Simonov, *Strong Hamburger moment problem*, Uch. Zapiski TNU **15** (2002), no. 1, 36–38.

Department of Mathematics, Donetsk National University, 24 Universitetskaya, 83055 Donetsk, Ukraine

*E-mail:*kirill_simonov@mail.ru