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ON SOME FUNCTIONAL EQUATIONS OF ADDITION THEOREM TYPE

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1. Addition theorems of rational type. We will discuss functional equations of the type

$$f(t+s)=\frac{\sum_{i=1}^{n} y_i(t)u_i(s)}{\sum_{j=1}^{m} z_j(t)v_j(s)}. \quad (1)$$

By a solution of (1) we mean a function f for which there exist functions y_i, u_i, z_j, v_j satisfying (1); thus we actually speak about functions that admit an "addition theorem"of rational type. Obviously, we may assume that the functions y_i (as well as u_i, z_j, v_j) are linearly independent.

It should be noted that equation (1) arises in a wide variety of situations. Let us consider how such equations arise in the context of integrable systems of particles on the line. Let $q_1(t), q_2(t), ..., q_n(t)$ be the coordinates of N particles on the line, interacting with the integrable potential $\sum_{k=1}^{N} U(q_j - q_k)$. Then the dynamics of the systems is describing by the system of ODE

$$\ddot{q}=\sum_{k=1}^{N} U(q_j-q_k), \qquad j=1,2,...,n. \quad (2)$$

We say that a dynamical system admits a Lax representation if it is equivalent to the matrix equation $\dot{L}=[L,M]$, where L and M are matrix-valued functions. It follows from this representation that the $J_k=\frac{1}{k}tr\{L_k\}$ $\quad (k=1,2,...,n)$ are integrals of the system (2). If it is proved that they are independent and in involution, then the system is completely integrable.

So starting with the ansatz for the matrices L and M one seeks restrictions necessary to obtain equations of motion (2). These restrictions typically involve the study of functional equations. For example, beginning with the ansatz

$$\begin{cases} L_{jk}=p_j\delta_{jk}+g(1-\delta_{jk})A(q_j-q_k), \\ M_{jk}=g\left[\delta_{jk}\sum_{l\neq k} B(q_j-q_l)-(1-\delta_{jk})C(q_j-q_k)\right] \end{cases}$$

one finds that $\dot{L}=[L,M]$ yields the equations of motion (2) for Hamiltonian system

$$H=\frac{1}{2}\sum_{j=1}^{N} p_j^2+g^2\sum_{j<k} U(q_j-q_k), \qquad U(x)=A(x)A(-x)+const$$

provided that $C(x)=-A'(x)$ and that $A(x)$ and $B(x)$ satisfy the functional equation

$$A(x+y)=\frac{A(x)A'(Y)-A'(x)A(y)}{B(x)-B(y)}. \quad (3)$$

In this sense the functional equation

$$\phi(t+s)=\frac{\alpha(t)\alpha'(s)-\alpha'(t)\alpha(s)}{\beta(t)\beta'(s)-\beta'(t)\beta(s)} \quad (4)$$

is said to be associated with the Lax approach to complete integrability and to the relativistic Calogero-Moser systems.

We will not concentrate on other examples, but note that in all works only analytic solutions of (1) were sought.

2. Reduction to the system of ODEs. Our approach to analysis of such functional equations is based on the reduction of (1) to an overdetermined system of ordinary differential equations. We assume that all functions y_i and z_j are continuously differentiable on some interval $I_1 \in \mathbb{R}$, all functions u_i and v_j are continuously differentiable on some interval $I_2 \in \mathbb{R}$, and the function f is continuously differentiable on some interval $I \subset (I_1 + I_2)$. Let us introduce two notions.

Definition 1. We say that the families $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ are jointly linearly independent if the family $\{y_i z_j\}_{i,j}$ is linearly independent.

Definition 2. We say that the families $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ are jointly quadratically dependent if they satisfy the following nontrivial relation

$$\sum_{i,l=1}^{n} \sum_{j,k=1}^{m} C_{lk}^{ij} y_i(t) z_j(t) y_l(t) z_k(t) = 0$$

where C_{lk}^{ij} are constants.

Theorem 1. *Let families of functions $\{u_i\}$ and $\{v_j\}$ be jointly linearly independent, and let the same be true for families $\{y_i\}$ and $\{z_j\}$. Then functional equation (1) holds for some f if and only if there exist constants C_{lk}^{ij} such that*

$$\begin{cases} y_i' z_j - y_i z_j' = \sum_{l,k} C_{lk}^{ij} y_l z_k, & i \le n; j \le m, \\ u_i' v_j - u_i v_j' = \sum_{l,k} C_{ij}^{lk} u_l v_k, & i \le n; j \le m \end{cases} \tag{5}$$

To demonstrate this approach let us consider a functional equation

$$\frac{f(x+y)}{f(x-y)} = \frac{g(x)+g(y)}{g(x)-g(y)} \tag{6}$$

which was introduced by P. McGill [3] in the work on Brownian motion. He assumed f and g to be meromorphic and found six pairs (f, g) of solutions [4]:

(a) $g(z) = Az$,
(b) $g(z) = A \sin z$,
(c) $g(z) = A \tan z$,
(d) $g(z) = Asn(z; m)$,
(e) $g(z) = Asd(z; m)$,
(f) $g(z) = Asc(z; m)$.

The functions f related to these functions g are found with the aid of the formula $g'(z)/g(z) = 2f'(0)/f(2z)$.

In this work we will look for solutions of (6) in a wider class of functions having two derivatives on some interval of real axis. It will be shown that the general solutions of the equation is

$$f(z) = C(ds(\varepsilon z; k) - cs(\varepsilon z; k)), \qquad g(z) = Bsn(\varepsilon z; k). \tag{7}$$

All McGill's solutions are special or limiting cases of (7).

By differentiating, equation (6) can be easily reduce to the form (1):

$$\frac{f'(x+y)}{f(x+y)} = \frac{g(x)g'(y) - g'(x)g(y)}{g^2(x) - g^2(y)}. \tag{8}$$

It is easy to see that $g(0) = 0$ and $g'(0) \ne 0$ since $f'(x)/f(x) = g'(0)/g(x)$ and f is not constant function. Writing system of differential equations (5) for (8) and taking into account jointly linear independence of the families $\{g, g'\}$ and $\{g^2, 1\}$, we conclude that $c_{21}^{11} = 1, c_{22}^{12} = -1, c_{12}^{22} = -c_{11}^{21}$ and all others coefficients are equal to zero. Thus, the system (5) has the form

$$\begin{cases} g'' = ag + cg^3 \\ g''g^2 - 2g(g')^2 = bg - ag^3 \end{cases}$$

where $a = c_{11}^{21}, b = c_{12}^{21}, c = c_{11}^{21}$. The last system is equivalent to the equation

$$(g')^2 = \frac{c}{2}g^4 + ag^2 - \frac{b}{2}.$$

Note that $b \neq 0$ since $g'(0) = 0$ and $g(0) = 0$. So, for $y = g\sqrt{-2/b}$ we obtain the differential equation

$$(y')^2 = 1 + ay^2 - \frac{bc}{4}y^4.$$

Its solution is $y(x) = sn(\varepsilon x; k)/\varepsilon$ where $(1 + k^2)\varepsilon^2 = -a, k^2\varepsilon^4 = -bc/4$. Thus,

$$g(x) = \sqrt{-\frac{b}{2}}\,\frac{sn(\varepsilon x; k)}{\varepsilon}. \tag{9}$$

>From the Galley addition theorem for Jacobian elliptic sinus and from (8) we get

$$(\ln f(x))' = \varepsilon/sn(\varepsilon x; k),$$

and consequently (see, for example [9])

$$f(x) = C(ds(\varepsilon x; k) - cs(\varepsilon x; k)). \tag{10}$$

Direct calculations show, that the pair (f, g) obtained by formulas (10) and (9), satisfy the equation (6).

Now it is easy to see that McGill's solutions are particular cases of (10)- (9), corresponding to the following values of parameters k and ε:

(a) $k \to 0$, $\varepsilon \to 0$,
(b) $k \to 0$,
(c) $k = 1$, $\varepsilon = \imath$,
(e) $k = m/(m-1)$, $\varepsilon = \sqrt{1-m}$,
(f) $k = 1 - m$, $\varepsilon = \imath$.

3. The general solution of (1). Basing on the Proposition 1 we can obtain description of the class of functions admitting addition theorem (1). Namely, the following theorem holds.

Theorem 2. *Let f be a function satisfying (1) and let the families of functions $\{u_i\}$ and $\{v_j\}$ be jointly linearly independent. Then, unless the families $\{y_i\}$ and $\{z_j\}$ are jointly quadratically dependent, all y_i and z_j are quasipolynomials up to a common multiplier. Function f itself is also a ratio of quasipolynomials.*

Thus "degenerated"situations are the most interesting ones. Let us consider very important for applications "symmetric"case $m = n = 2$

$$f(t+s) = \frac{y(t)u(s) - u(t)y(s)}{z(t)v(s) - v(t)z(s)}, \tag{11}$$

which includes (4) and (3) and was solved by Buchstaber [1] for analytic f. It is proved in [7], that removing the assumption of analyticity doesn't change the form of solution:

$$F(x) = Ce^{\lambda x}\frac{\Phi(x;\nu_1}{\Phi(x;\nu_2)}, \qquad where \Phi(x;\nu) = \frac{\sigma(\nu - x)}{\sigma(\nu)\sigma(x)}e^{\zeta(\nu)x}.$$

Here $\sigma(x)$ and $\zeta(x)$ are Weierstrass sigma and zeta functions, $\Phi(x;\nu)$ is Baker-Akhiezer function.

4. The Levi-Civita functional equation. If we put $m = 1$ in the equation (1) we lead to Levi-Civita functional equation

$$f(t+s) = \sum_{i=1}^{n} y_i(t)u_i(s) \tag{12}$$

which was originally considered in full generality by Levi-Civita (1913), Stephanos (1904) and Stäkel (1913). It appeared in boundary-value problems of mathematical physics and was solved with the assumption that all unknown functions to be n-times differentiable. In this assumption (12) may be reduced to a homogeneous linear differential equation of n-th order with constant coefficients. Thus the general solution of (12) is a quasipolynomial of n-th order:

$$f(t) = \sum_{k=1}^{m} P_k(t)e^{\lambda_k t}, \qquad \sum_{k=1}^{m}(degP_k + 1) = n.$$

Later on the theory of Levi-Civita equations was extended in various directions. Some interesting problems arise when solutions are considered on general groups or semigroups or on an arbitrary, possibly thin, subsets of the real axis. The most general result for the Levi-Civita equation on semigroup is the following [5]:

Theorem 3. *Let G be a semigroup with the unit and let $f : G \to C$ be a function such that*

$$f(gh) = \sum_{i=1}^{n} y_i(g)u_i(h) \qquad \forall g, h \in G.$$

Then f is a matrix element of n-dimensional representation of G. For locally compact group G the representation can be chosen preserving such properties of solution as boundedness and continuity.

(By a matrix element of representation T of semigroup G in linear space X we mean a function

$$f(g) = \langle T(g)\xi, \eta\rangle,$$

where $\xi \in X, \eta \in X^*$.)

The solutions of Levi-Civita equation have a simple geometric characterization: they are the functions whose orbits under the regular representation belong to finite-dimensional subspaces. Basing on this fact it is possible to prove [6] that Levi-Civita equation are stable when considered in various functional classes on amenable groups. (We mean the Hyers - Ulam concept of stability: an equation is stable when each function "almost satisfying"it is "close"to a proper solution.) The proof relies on the theory of covariant n-widths (that is distances from an invariant convex set to invariant n-dimensional subspaces) which was developed in the same work.

5. Addition theorems in three variables. In conclusion we will note that solution of the Levi-Civita equation may be interpreted as functions for which $f(x+y)$ belongs to algebra of functions generating by functions of one variable. This problem can be generalized for the case of n variables: to describe all functions $f(t)$ such that $f(x_1 + x_2 + ... + x_n)$ belongs to algebra of functions which depend of less than n variables. As a first step one can consider a functional equation of three variables

$$f(x+y+z) = a_1(x)b_1(y,z) + a_2(y)b_2(x,z) + a_3(z)b_3(x,y). \tag{13}$$

Particular cases of (13) were studied in a lot of works (see, for example, [2]-[8]) and have many applications. Investigation of such functional equations 'so lead to some interesting semigroup problems.

We will mention here only one result on this subject:

Theorem 4. . *Let f be a continuous complex-valued function admitting an addition theorem (13). Then f is a quasipolynomial.*

REFERENCES

[1] H. W. Braden, V. M. Buchstaber, *The general analytic solution of a functional equation of addition type*, SIAM J. Math. Anal. 28 (1997), no. 4, 903-923.
[2] Bae Jae-Hyeong, Jun Kil-Woung, ..., Bull. Korean Math. Soc. 38 (2001), no. 2, 325–336.
[3] P. McGill, *Wiener-Hopf factorization of Brownian motion*, Probab. Theory Related Fields 83 (1989), 355–390.
[4] P. McGill, *Jacobi elliptic functions and change of variable in a convolution*, Aequation Math. 39 (1990), 114–119.
[5] E. Shulman, *Functional equations of homological type*, Ph. D. Thesis, Moscow, 1994.
[6] E. Shulman, *Group representations and stability of functional equations*, J. London Math. Soc., 54 (v.2), 1996, 111–120.
[7] E. Shulman, *On rational addition theorems*, to appear in J. Math. Anal. Appl.
[8] Trif T., *Hyiers-Ulam-Rassias stability of a Jensen type functional equations*, J. Math. Anal. Appl. 250 (2000), no. 2, 579–588.
[9] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Univ. Press, Cambridge, 1952.

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