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INVARIANT AND HYPERINVARIANT SUBSPACES OF THE OPERATOR J^α IN THE SOBOLEV SPACES

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1. Introduction

It is well known ([3], [5], [9]) that the Volterra integration operator defined on $L_p[0,1]$ by $J\colon f \to \int_0^x f(t)\,dt$ is unicellular for $p \in [1,\infty)$ and its lattice of invariant subspaces is anti-isomorphic to the segment $[0,1]$.

The same is also true for the complex powers of the integration operator J:

$$J^\alpha\colon f \to \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt, \quad \operatorname{Re}\alpha > 0. \tag{1}$$

More precisely, the lattice of invariant and hyperinvariant subspaces of the operator J^α are of the form:

$$\operatorname{Lat} J^\alpha = \operatorname{Hyplat} J^\alpha = \{E_a := \chi_{[a,1]} L_p[0,1]\colon 0 \le a \le 1\}. \tag{2}$$

E. Tsekanovskii [12] has obtained a description of the lattice $\operatorname{Lat} J_k$ of invariant subspaces of the integration operator J_k defined on the Sobolev space $W_2^k[0,1]$.

I. Domanov and M. Malamud [4] have described the lattices $\operatorname{Lat} J_k^\alpha$ and $\operatorname{Hyplat} J_k^\alpha$ of invariant and hyperinvariant subspaces of the operator J_k^α defined on $W_p^k[0,1]$ and investigated the operator algebras $\operatorname{Alg} J_k^\alpha$, commutant $\{J_k^\alpha\}'$, and double commutant $\{J_k^\alpha\}''$.

In particular, it is shown in [4] that the operator J^α is unicellular on $W_p^k[0,1]$ (with $k \ge 2$) if and only if $\alpha = 1$.

It is also shown in [4] that $\operatorname{Hyplat} J_k^\alpha = \operatorname{Hyplat} J_k = \operatorname{Hyplat}^c J_k \cup \operatorname{Hyplat}^d J_k$, where

$$\operatorname{Hyplat}^c J_k = \{E_a\colon 0 \le a \le 1\}, \qquad E_a := \{f\colon f \in W_p^k[0,1], f = 0 \text{ for } x \in [0,a]\} \tag{3}$$

is a continuous chain and $\operatorname{Hyplat}^d J_k = \{E_l^k\}_{l=0}^k$ with $E_k^k := W_p^k[0,1]$ and

$$E_l^k = \{f \in W_p^k[0,1]\colon f(0) = \cdots = f^{(k-l-1)}(0) = 0\}, \qquad l \in \{0,1,\ldots,k-1\} \tag{4}$$

is a discrete chain.

In [11] several results from [4] has been generalized to the case of Liouville spaces $L_p^s[0,1]$ $(s > 0)$.

In the paper under consideration we extend some results mentioned above to the case of the Sobolev spaces $W_p^s[0,1]$ $(s > 0)$. For integer $s = k \in \mathbb{Z}_+$ our results coinside with that from [4]. We apply and generalize the method proposed in [4].

Note, that the description of the lattices $\operatorname{Lat} J_s^\alpha$ and $\operatorname{Hyplat} J_s^\alpha$ depends on the embedding index of the space $W_p^s[0,1]$ in $C^k[0,1]$. For instance, the lattices $\operatorname{Lat} J_s^\alpha$ and $\operatorname{Lat} J_k^\alpha$ coinside $(k = [s],\ s - k \le 1/p)$.

2. Notations

$W_p^k[0,1]$ stands for the Sobolev space: $f \in W_p^k[0,1]$ if f has $k-1$ absolutely continuous derivatives and $f^{(k)} \in L_p[0,1]$.

Let $s \in \mathbb{R}_+$, $s = [s] + \varepsilon$. $W_p^s[0,1]$ $(1 \le p < +\infty)$ stands for the Sobolev space: $f \in W_p^s[0,1]$ if $f \in W_p^{[s]}[0,1]$ and for derivative $f^{([s])}$ of the order $[s]$ is fulfilled

$$\langle f^{([s])}\rangle_{p,\varepsilon} = \left(\int_0^1 dx \int_0^1 \frac{|f^{([s])}(x) - f^{([s])}(y)|^p}{|x-y|^{1+p\varepsilon}}\, dy\right)^{1/p} < +\infty.$$

Let $C_0^\infty[0,1] = \{f \in C^\infty[0,1] : f^{(j)}(0) = 0,\ j \in \mathbb{Z}_+\}$. We denote by $W_{p,0}^s[0,1]$ the closure of the lineal $C_0^\infty[0,1]$ in $W_p^s[0,1]$.

3. Cyclic subspaces of the operator J^α

Let $J_{s,0}^\alpha$ and J_s^α stand for the operator J^α defined on $W_{p,0}^s[0,1]$ and $W_p^s[0,1]$ respectively. In what follows we assune that either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\operatorname{Re}\alpha > k - \frac{1}{p}$. Under this assumption the operator J_s^α is well defined on $W_p^s[0,1]$.

Definition 1. ([9]) Recall that a subspace E of a Banach space X is called a cyclic subspace for an operator $T \in [X]$ if $\operatorname{span}\{T^n E : n \ge 0\} = X$.
The set of all cyclic subspaces of an operator T is denoted by $\operatorname{Cyc}(T)$.
A vector $f \in X$ is called a cyclic vector if $E := \{\lambda f : \lambda \in \mathbb{C}\} \in \operatorname{Cyc}(T)$

$\mu_T := \inf_E\{\dim E : E \in \operatorname{Cyc}(T)\}$ is called the spectral multiplicity of an operator T in X.

An operator T is called cyclic if $\mu_T = 1$.

Lemma 1. *Let $\operatorname{Re}\alpha > 0$. Then the operator $J_{s,0}^\alpha$ is cyclic. Moreover the following equivalence holds:*

$$f \in \operatorname{Cyc} J_{s,0}^\alpha \iff \int_0^\varepsilon |f(x)|^p\, dx > 0 \quad \textit{for all} \quad \varepsilon > 0. \tag{5}$$

In what follows we denote

$$k = \begin{cases} [s], & s - [s] \le 1/p, \\ [s]+1, & s - [s] > 1/p. \end{cases} \tag{6}$$

Now we present a description of the cyclic subspaces of the operator J_s^α.

Proposition 1. 1) The spectral multiplicity $\mu_{J_s^\alpha}$ of J_s^α is

$$\mu := \mu_{J_s^\alpha} = \begin{cases} \min\{\alpha, k\}, & \alpha \in \mathbb{Z}_+ \setminus \{0\}, \\ k, & \alpha \notin \mathbb{Z}_+ \setminus \{0\}. \end{cases} \tag{7}$$

2) The system $\{f_j\}_1^N$ of vectors $f_j \in W_p^s[0,1]$ generates a cyclic subspace for J_s^α if and only if:

i) $N \ge \mu$

ii) $\operatorname{rank} W_\mu\{f_1, \dots, f_N\}(0) = \mu$, where

$$W_\mu\{f_1,\dots,f_N\}(x) = \begin{pmatrix} f_1(x) & \dots & f_N(x) \\ f_1'(x) & \dots & f_N'(x) \\ \dots & \dots & \dots \\ f_1^{(\mu-1)}(x) & \dots & f_N^{(\mu-1)}(x) \end{pmatrix}$$

We note (see the embedding theorem [1]), that the space $W_p^s[0,1]$ is continuously embedded to the space $C^{[s]}[0,1]$ for $\{s\} > 1/p$. Hence the codimention of the subspace $W_{p,0}^s[0,1]$ in $W_p^s[0,1]$ is equal $\dim W_p^s[0,1]/W_{p,0}^s[0,1] = \begin{cases} [s], & \text{for } \{s\} = s - [s] \le 1/p, \\ [s]+1, & \text{for } \{s\} = s - [s] > 1/p. \end{cases}$

Example 1. The system $\{f_1 = 1 + x^{10}, f_2 = x + x^{20}\}$ generates a cyclic subspace in $W_p^s[0,1]$ for J_s^3 if $1 + 1/p < s \le 2 + 1/p$ and does not generate it if $s > 2 + 1/p$.

4. Invariant subspaces of the operator J^α

Definition 2. Let X be a Banach space. An operator $T \in [X]$ is called unicellular if its lattice of invariant subspaces $\operatorname{Lat} T$ is linearly ordered.

Lemma 2. *Let* $\operatorname{Re}\alpha > 0$. *Then*

$$\operatorname{Lat} J_{s,0}^\alpha = \{E_a^s : 0 \le a \le 1\}, \quad E_a^s = \{f \in W_{p,0}^s[0,1] : f(x) = 0 \text{ for } x \in [0,a]\} \tag{8}$$

and thus $J_{s,0}^\alpha$ *is unicellular.*

Our description of $\operatorname{Lat} J_s^\alpha$ is based on the description of the lattice $\operatorname{Lat} Q$ for a nilpotent operator $Q \in [\mathbb{C}^k]$ obtained by L. Brickman and P. A. Fillmore [2].

For each bounded operator T defined on a Banach space X, ($T \in [X]$) and $E \in \operatorname{Lat} T$ we denote by $\hat{T}_E$ the quotient operator acting on the quotient space X/E according to the natural rule $\hat{T}\hat{f} = \widehat{(Tf)}$, where $\hat{f}$ stands for a coset $\hat{f} := f + E$.

Theorem 1. *Let* π *be the quotient map,*

$$\pi \colon W_p^s[0,1] \to X_k := W_p^s[0,1]/W_{p,0}^s[0,1],$$

and $\hat{J}_s^\alpha$ *be the quotient operator on* X_k, *where* k *is defined by (6).*

Then $\operatorname{Lat} J_s^\alpha = \operatorname{Lat}^c J_s^\alpha \cup \operatorname{Lat}^d J_s^\alpha$, *where*

1)

$$\operatorname{Lat}^c J_s^\alpha = \{E_a^s \colon 0 \le a \le 1\}, \qquad E_a^s = \{f \in W_{p,0}^s[0,1] \colon f(x) = 0 \text{ for } x \in [0,a]\} \tag{9}$$

is a "continuous part" of $\operatorname{Lat} J_s^\alpha$;

2)

$$\operatorname{Lat}^d J_s^\alpha = \pi^{-1}(\operatorname{Lat} \hat{J}_s^\alpha) = \bigcup_M \pi^{-1}\{[M, (\hat{J}_s^\alpha)^{-1}M] \colon M \in \operatorname{Lat}(\hat{J}_s^\alpha | \hat{J}_s^\alpha M)\} \tag{10}$$

is a "discrete part" of $\operatorname{Lat} J_s^\alpha$.

Here $[M, (\hat{J}_s^\alpha)^{-1}M]$ *is a closed interval in the lattice of all subspaces of* X_k. *Each interval satisfies the equation*

$$\dim(\hat{J}_s^\alpha)^{-1}M - \dim M = d,$$

where $d = \min\{-[-\alpha], k\}$.

Corollary 1. *Let* $0 < s \le 1 + 1/p$ *and* $\operatorname{Re}\alpha > s - \frac{1}{p}$. *Then*

$$\operatorname{Lat} J_s^\alpha = \operatorname{Lat} J_{s,0}^\alpha \cup W_p^s[0,1] = \{E_a : 0 \le a \le 1\} \cup W_p^s[0,1].$$

In particular, the operator J_s^α *is unicellular in* $W_p^s[0,1]$.

Corollary 2. *Let* $s > 1 + 1/p$ *and either* $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ *or* $\operatorname{Re}\alpha > s - \frac{1}{p}$.

Then: 1) the operator J_s^α *is unicellular in* $W_p^s[0,1]$ *if and only if* $\alpha = 1$;

2) $\operatorname{Lat} J_s = \operatorname{Lat}^c J_s \cup \operatorname{Lat}^d J_s$, *where* $\operatorname{Lat}^c J_s$ *is defined by* (9) *and*

$$\operatorname{Lat}^d J_s = \{W_{p,0}^s[0,1] = E_0^s \subset E_1^s \subset \cdots \subset E_k^s := W_p^s[0,1]\},$$

where

$$E_l^s := \operatorname{span}\{W_{p,0}^s[0,1], x^{k-1}, \dots, x^{k-l}\}, \quad l \in \{1, \dots, k\}.$$

Here k is defined by (6).

5. Commutant $\{J_s^\alpha\}'$ and the lattice Hyplat J_s^α

As usual $\{T\}'$ stands for the commutant of the operator $T \in [X]$ defined on a Banach space X: $\{T\}' = \{R \in [X]: RT = TR\}$. A subspace $E \in \operatorname{Lat} T$ is called a hyperinvariant subspace for T if E is invariant for any $R \in \{T\}'$.
Hyplat T stands for the lattice of all hyperinvariant subspaces of T.
We will say that the lattice Hyplat T is unicellular if it is linearly ordered.

Proposition 2. Let $\operatorname{Re}\alpha > 0$. Then $R \in \{J_{s,0}^\alpha\}'$ if and only if R is bounded and

$$(Rf)(x) = \frac{d}{dx}\int_0^x r(x-t)f(t)\,dt, \qquad r(x) \in L_{p'}[0,1], \quad (p')^{-1} + p^{-1} = 1.$$

Corollary 3. *The lattices of invariant and hyperinvariant subspaces of the operator $J_{s,0}^\alpha$ coinside:*

$$\operatorname{Hyplat} J_{s,0}^\alpha = \operatorname{Lat} J_{s,0}^\alpha = \{E_a^s: 0 \le a \le 1\}, \qquad E_a^s = \{f \in W_{p,0}^s[0,1]: f(x) = 0, x \in [0,a]\}.$$

Now we present a description of the commutant $\{J_s^\alpha\}'$.

Theorem 2. *Let either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\operatorname{Re}\alpha > s - \frac{1}{p}$. Then $R \in \{J_s^\alpha\}'$ if and only if*

$$(Rf)(x) = \frac{d}{dx}\int_0^x r(x-t)f(t)\,dt, \qquad r(x) \in W_p^s[0,1].$$

In particular, $\{J_s^\alpha\}'$ is a commutative algebra and does not depend on α.

To prove the theorem we consider the block-matrix representation of the operator J_s^α with respect to the direct sum decomposition $W_p^s[0,1] = W_{p,0}^s[0,1] \dot{+} X_k$ (see [4]).

Remark 1. For $s = 0$, that is for the space $L_p[0,1] =: W_p^0[0,1]$, Proposition 2 and Theorem 2 have been obtained by M. Malamud in [8].

Corollary 4. *The double commutant $\{J_s^\alpha\}''$ of the operator J_s^α coinsides with its commutant: $\{J_s^\alpha\}'' = \{J_s^\alpha\}' = \{J_s\}' = \{J_s\}''$.*

Combining Corollary 2 with Corollary 4 we easily get

Proposition 3. The lattice Hyplat J_s^α is unicellular, that is Hyplat $J_s^\alpha = \operatorname{Lat} J_k$.

Combining Corollary 1 with Corollary 4 we arrive at the following cours

Corollary 5. *Let $\alpha \ne 1$. Then* Hyplat $J_s^\alpha = \operatorname{Lat} J_s^\alpha$ *if and only if* $0 \le s \le 1 + 1/p$.

Corollary 6. *Let $\alpha \in \mathbb{Z}_+ \setminus \{0\}$, $k-1 < s \le k$, $\alpha \le k$. Then*

$$\operatorname{Hyplat}^d J_s^\alpha = \pi^{-1}(\operatorname{Hyplat} J^\alpha(0;k))$$

if and only if k is odd and $\alpha = 2$.

Example 2. Let $3 + 1/p < s \le 4 + 1/p$, $\alpha = 2$ and $J_s^2: W_p^s[0,1] \to W_p^s[0,1]$. Then $\operatorname{Hyplat}^d J_s^2 = \operatorname{Lat}^d J_s^2 = \{E_0^s, E_1^s, E_2^s, E_3^s, E_4^s\}$, but $\pi^{-1}(\operatorname{Hyplat} J^2(0;s)) = \{E_0^s, E_2^s, E_4^s\}$.

Theorem 2 allows us to present a description of the algebra Alg J_s^α.

Proposition 4. The following are true:

1) If either $\alpha = 1$ or $s \le 1 + 1/p$, then $\operatorname{Alg} J_s^\alpha = \{J_s^\alpha\}''$;

2) If $1 + 1/p < \alpha \le s$, then $\operatorname{Alg} J_s^\alpha = \{T = cI + R\colon c \in \mathbb{C}, R \in \operatorname{Alg}_0 J_s^\alpha\}$, where

$$\operatorname{Alg}_0 J_s^\alpha = \{R\colon Rf = r * f, \quad r \in W_p^{s-1}[0,1], \quad r^{(j)}(0) = 0 \quad \text{for } j \ne i\alpha - 1, \quad i \le \left[\frac{s}{\alpha}\right]\};$$

3) If $s > 1 + 1/p$ and $\operatorname{Re}\alpha > s - \frac{1}{p}$, then

$$\operatorname{Alg} J_s^\alpha = \{T = cI + R\colon c \in \mathbb{C}, \quad Rf = r * f, \quad r \in W_{p,0}^{s-1}[0,1]\}.$$

Corollary 7. *Let T be a bounded operator on $W_p^s[0,1]$ and $\hat{T}$ be a quotient operator defined on $\mathbb{C}^k \cong W_p^s[0,1]/W_{p,0}^s[0,1]$, where k is defined by (6). Then $T \in \operatorname{Alg} J_s^\alpha$ if and only if the quotient operator*

$$\hat{T} \in \operatorname{Alg}(J(0;s)^\alpha) = \{J(0;s)^\alpha\}''.$$

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