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SOME EQUIVALENT QUASINORMS ON OPERATOR IDEALS

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Some special quasi-norms on the s-number ideals $L_\phi(E,F)$, [5], [6], [11], are considered. The special case of the entropy numbers is also studied.

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1. Introduction Let $T \in L(X)$ be a linear and bounded operator $T : X \to X$, where X is a Banach space. The sequence of the approximation numbers $(a_n(T))$ is defined as follows:

$$a_n(T) = inf\{||T - A|| : A \in L(X) \quad rankA < n\}, n = 1, 2 \dots.$$

If X is a Hilbert space and $T \in L(X)$ is compact, the sequence $(a_n(T))$ coincides with the sequence $\{\lambda_n(TT^*)^{\frac{1}{2}}\}$, where $\lambda_n(T)$ is the sequence of the eigenvalues of T, ordered in a convenient way [2],[5],[11]. Are also other s-number sequences for an operator $T \in L(X)$, [10], [11]. For example the Kolmogorov numbers $d_n(T) = inf\{||Q_N^X T|| : N \subset X$ and $dim N < n,\}$ where Q_N^X is the canonical surjection from X onto X/N.

In the following by $(s_n(T))$ will be denoted an additive s - number sequence $(s_{2n-1}(S + T) \le s_n(S) + s_n(T),\ n = 1, 2, \dots)$. The approximation and the Kolmogorov numbers are additive. We recall that the sequence $(s_n(T))$ is such that $||T|| = s_1(T) \ge s_2 \ge \dots \ge 0$. Let l_∞ be the space of all bounded real sequences. For $x \in l_\infty$, card(x) = card$\{i \in N : x_i \neq 0\}$. Let $K \subset l_\infty$ be the set of all sequences x such that: card$(x) \le n$ and $x_1 \ge x_2 \ge \dots \ge 0$.

A function $\phi : K \to R$ is called symmetric norming function [2], [4], [5], [6], [11], if:

$$\phi > 0 \quad for \quad all \quad x \neq 0 \tag{1}$$

$$\phi(\alpha x) = \alpha\phi(x), \quad x \in K, \alpha \ge 0 \tag{2}$$

$$\phi(x + y) \le \phi(x) + \phi(y) \tag{3}$$

$$\phi(1, 0, 0, \dots) = 1 \tag{4}$$

$$If \sum_1^k x_i \le \sum_1^k y_i, \quad k = 1, 2, \dots, then \quad \phi(x) \le \phi(y). \tag{5}$$

Example of such functions are

$$\phi_\infty(x) = x_1, \phi_1(x) = \sum_1^n x_i; \phi_\omega(x) = \sum_1^n \frac{x_i}{i},$$

[2],[5],[11]. It is known, [3], [11], that for all symmetric norming functions ϕ, the functions $\phi_{(p)} : (x_i) \in K \to (\Phi(\{x_i^p\}))^{\frac{1}{p}}, \quad 1 \le p < \infty$, are also symmetric norming functions.

If $x \in l_\infty$ and $x_1 \ge x_2 \ge \dots \ge 0$, we take

$$\Phi(x) = \lim_{n\to\infty} \Phi(x_1, \dots, x_n, 0, 0, \dots).$$

By means of the symmetric norming function and the sequence $(s_n(T)$, the class $L_\phi(X)$ is defined [5], [10], as follows

$$L_\Phi(X) = \{T \in L(X) : \Phi(\{s_n(T)\}) < \infty\}.$$

Because $s_{2n-1}(T_1+T_2) \leq s_n(T_1)+s_n(T_2), n = 1, 2, \ldots$ and $s_n(\alpha T) = |\alpha| s_n(T), \alpha$ beeing a scalar, from the properties of the function ϕ, it results that $\| T \|_\phi = \Phi(\{s_n(T)\})$ is a quasi-norm.

We remark that, more generaly, $s_{m+n-1}(T_1 + T_2) \leq s_n(T_1) + s_n(T_2)$, [3],[6],[11].

It is obvious that $\| T \|_\phi \geq 0$ and $\| \alpha T \|_\phi = |\alpha| \cdot \| T \|_\phi$. The relation $\| T_1 + T_2 \|_\phi \leq 2(\| T_1 \|_\phi + \| T_2 \|_\phi$ results from the properties (2), (3), (5) of the functions ϕ and the fact that

$$\sum_1^k s_n(T_1 + T_2) \leq 2 \sum_1^k (s_n(T_1) + s_n(T_2)), k = 1, 2, \ldots.$$

The last inequality is a simple consequence of the relation $s_{2n-1}(T_1 + T_2) \leq s_n(T_1) + s_n(T_2)$ since

$$\sum_1^k s_n(T_1 + T_2) \leq \sum_1^{2k} s_n(T_1 + T_2) = \sum_1^k s_{2n-1}(T_1 + T_2) + \sum_1^k s_{2n}(T_1 + T_2) \leq$$

$$\leq 2 \sum_1^k s_{2n-1}(T_1 + T_2).$$

If X is a Hilbert space, $\| T \|_\phi$ is a norm [2], [3], [6]. The study for the case of Banach spaces is made in [5], [10], [11]. In the following we present some equivalent quasinorms on $L_\phi(X)$.

2. Equivalent quasi-norms on $L_\phi(X)$

Firstly we present some well-known results,[6],[11]:

Proposition 1.1 *The quasi-norm* $\| T \|_\phi^+ = \Phi(\{s_{2n-1}(T)\})$ *is equivalent with* $\| T \|_\phi$.

The equivalence results from the fact that:

$$\sum_1^k s_{2n-1}(T) \leq \sum_1^k s_n(T) \leq 2 \sum_1^k s_{2n-1}(T), k = 1, 2, \ldots; T \in L(X).$$

The left inequality is a consequence of the fact that $(s_n(T))$ is decreasing and the right inequality is presented above.

Proposition 1.2. *The quasinorm* $\| T \|_{\phi_{(p)}}$ *is a equivalent with* $\| T \|_{\phi_{(p)}}^\nabla = \phi_{(p)} \left(\{ \frac{1}{n} \sum_{i=1}^n s_i(T) \} \right)$, $1 < p < \infty$.

This is a consequence of the Hardy's inequality, namely:

$$\sum_{n=1}^k s_n^p(T) \leq \sum_1^k \left(\frac{1}{n} \sum_{i=1}^n s_i(T)^p \right) \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^k s_n^p(T), \quad 1 < p < \infty$$

and the properties of the functions Φ.

Now we generalize these results as follows:

Proposition 1.3 *The quasinorm* $\|T\|_\Phi^* = \Phi\left(s_{nk-(k-1)}(T)\right)$ *is equivalent with* $\|T\|_\Phi$, *for all* $k \geq 2$.

Proof. For $k = 2$ we obtain proposition 1.1. For all $k \geq 3$, because $(s_n(T))$ is decreasing, we can write:

$$\sum_{n=1}^r s_{(n-1)k+1}(T) \leq \sum_1^r s_n(T) \leq \sum_{n=1}^{rk} s_n(T) =$$

$$= \sum_{n=1}^{r} \sum_{i=(n-1)k+1}^{nk} s_i(T) \leq k \sum_{n=1}^{r} s_{(n-1)k+1}(T), r = 1, 2, \ldots.$$

By using the properties (2) and (5) of the functions Φ, we obtain

$$\|T\|_\Phi^* \leq \|T\|_\Phi \leq k\|T\|_\Phi^*.$$

Theorem 1.4 *If the sequence (α_n) is such that $\alpha_1 \geq \ldots \geq \alpha_n \geq \ldots > 0$ and $\lim \alpha_n \neq 0$, then the quasinorm $\| T \|_{\phi_{(p)}}$ is equivalent with $\| T \|_{\phi_{(p)}}^{\circ} = \Phi_{(p)} \left(\{ \frac{1}{\alpha_1 + \ldots + \alpha_n} \sum_{i=1}^{n} \alpha_i s_i(T) \} \right)$, $1 < p < \infty$.*

Proof. Since the sequences (α_i) and $(s_i(T))$ are decreasing it results that

$$\frac{1}{n\alpha_1} n\alpha_n s_n(T) = \frac{\alpha_n}{\alpha_1} s_n(T) \leq \frac{1}{\alpha_1 + \ldots + \alpha_n} \sum_{i=1}^{n} \alpha_i s_i(T) \leq \frac{1}{n\alpha_n} \alpha_1 \sum_{1}^{n} s_i(T).$$

If $\lim \alpha_i = \alpha \neq 0$ we obtain:

$$\frac{\alpha}{\alpha_1} s_n(T) \leq \frac{1}{\alpha_1 + \ldots + \alpha_n} \sum_{i=1}^{n} \alpha_i s_i(T) \leq \frac{\alpha_1}{\alpha} \left(\frac{1}{n} \sum_{i=1}^{n} s_i(T) \right).$$

From the Hardy's inequality we obtain:

$$\sum_{n=1}^{k} \left(\frac{\alpha}{\alpha_1} s_n(T) \right)^p \leq \sum_{n=1}^{k} \left(\frac{1}{\alpha_1 + \ldots + \alpha_n} \sum_{i=1}^{n} \alpha_i s_i(T) \right)^p \leq \sum_{n=1}^{k} \left(\frac{\alpha_1}{\alpha} \right)^p \left(\frac{1}{n} \sum_{i=1}^{n} s_i(T) \right)^p \leq$$

$$\leq \left(\frac{\alpha_1}{\alpha} \right)^p \left(\frac{p}{p-1} \right)^p \sum_{1}^{k} s_n^p(T), \quad 1 < p < \infty.$$

From the properties of the functions Φ it results
$\frac{\alpha}{\alpha_1} \| T \|_{\phi_{(p)}} \leq \| T \|_{\phi_{(p)}}^{\circ} \leq \frac{\alpha_1}{\alpha} \frac{p}{p-1} \| T \|_{\phi_{(p)}}$, i. e., $\| T \|_{\phi_{(p)}}$ is equivalent with $\| T \|_{\phi_{(p)}}^{\circ}$.
Remark. For the particular case, $\alpha_n = \frac{1}{n}$, we obtain the proposition 1.2 and if Φ is Φ_1 : $(s_n(T)) \to \sum s_n(T)$ we obtain the results from [7].

The following result will be of great interest for applications. We remark that, for all ϕ, the function $\overline{\Phi} : (x_i) \in K \to \Phi(\{\epsilon_i x_i\})$ is also a symmetric norming function, if $1 = \epsilon_1 \geq \epsilon_2 \geq \ldots \geq 0$, [11].

Theorem 1.3 *The quasi-norm $\| \overline{T} \|_{\overline{\phi}} = \overline{\Phi}(\{s_{n^r}(T)\})$ is equivalent with $\| T \|_{\overline{\Phi}}$, $r \geq 2$ if exists a constant c such that $\epsilon_{n^r} \leq \frac{c}{n^{r-1}} \epsilon_n$, $\forall n \in N$.*

Proof. Firstly we remark that

$$\sum_{1}^{k} \epsilon_n s_{n^r}(T) \leq \sum_{1}^{k} \epsilon_n s_n(T), \quad k = 1, 2, \ldots,$$

because $(s_n(T))$ is decreasing.

Let now $j \in N$ such that $j^r \leq k < (j+1)^r$. Then we can write:

$$\sum_{1}^{k} \epsilon_n s_n(T) \leq \sum_{1}^{j} (2^r - 1) n^{r-1} \epsilon_{n^r} s_{n^r}(T) \leq c(2^r - 1) \sum_{1}^{k} \epsilon_n s_{n^r}(T).$$

From the properties (2), (5) of the functions ϕ, we obtain:

$$\Phi(\epsilon_n s_{n^r}(T)) \leq \Phi(\epsilon_n s_n(T)) \leq c(2^r - 1) \Phi(\epsilon_n s_{n^r}(T)).$$

Hence $\| \overline{T} \|_{\overline{\Phi}} \leq \| T \|_{\overline{\Phi}} \leq c(2^r - 1) \| \overline{T} \|_{\overline{\Phi}}$.

3. The special case of $L_{\infty,q,\gamma}(X)$ The class

$$L_{\infty,q,\gamma}(X)=\{T:\left(\sum_{1}^{\infty}[(1+\log n)^{\gamma}s_n(T)]^q\cdot n^{-1}\right)^{\frac{1}{q}}<\infty\}$$

has been studied in [2], [11], $0<q<\infty$, $-\frac{1}{q}<\gamma<\infty$. It is simple to prove that

$$\| T \|_{\infty,q,\gamma}=\left(\sum_{1}^{\infty}[(1+\log n)^{\gamma}s_n(T)]^q n^{-1}\right)^{\frac{1}{q}}$$

is a quasi-norm on $L_{\infty,q,\gamma}(X)$.

Remark. If the sequence $\{(1+\log n)^{\alpha_q}\cdot n^{-1}\}$ is decreasing, the function

$$\tilde{\phi}:(s_n(T))\to\left(\sum_{1}^{\infty}[(1+\log n)^{\gamma}s_n(T)]^q n^{-1}\right)^{\frac{1}{q}}$$

is a symmetric norming function, $1\leq q<\infty$, and hence the class $L_{\infty,q,\gamma}(X)$ is a special case of $L_{\phi}(X)$. For $\alpha=0$ we obtain the class

$$L_{\phi_{\omega(q)}}(X)=\{T:\left(\sum_{1}^{\infty}\frac{s_n^q(T)}{n}\right)^{\frac{1}{q}}<\infty\}.$$

Here we present some quasi-norms equivalent with $\| T \|_{\infty,q,\gamma}$ for all q,γ, $0<q<\infty$, $-\frac{1}{q}<\gamma<\infty$.

Theorem 2.1 *The quasi-norm* $\| T \|^{+}_{\infty,q,\gamma}=\left(\sum[(1+logn)]^{\gamma}s_{n^k}(T)]^q\cdot n^{-1}\right)^{\frac{1}{q}}$ *is equivalent with* $\| T \|_{\infty,q,\gamma}$.

Proof. Because $s_n(T)$ is decreasing it results that $\| T \|^{+}_{\infty,q,\gamma}\leq\| T \|_{\infty,q,\gamma}$.
On the other hand

$$\| T \|_{\infty,q,\gamma}=\left(\sum_{n=1}^{\infty}\sum_{i=n^k}^{(n+1)^k-1}[(1+\log i)^{\gamma}s_i(T)]^q i^{-1}\right)^{\frac{1}{q}}\leq$$

$$\leq\left(\sum_{1}^{\infty}(2^k-1)n^{k-1}[max\{(1+\log n^k)^{\gamma},(1+\log(n+1)^k)^{\gamma}\}s_{n^k}(T)]^q\cdot n^{-k}\right)^{\frac{1}{q}}\leq$$

$$\leq(2^k-1)\left(\sum_{1}^{\infty}c(\gamma,q,k)(1+\log n)^{\alpha q}s^q_{n^k}(T)\cdot n^{-1}\right)^{\frac{1}{q}}=\tilde{c}(\gamma,q,k)\,\| T \|^{+}_{\infty,q,\gamma}.$$

Hence

$$\| T \|^{+}_{\infty,q,\gamma}\leq\| T \|_{\infty,q,\gamma}\leq\tilde{c}(\gamma,q,k)\,\| T \|^{+}_{\infty,q,\gamma}.$$

Now we denote $\alpha_n(T)=s_{2^{n-1}}(T), n=1,2,\ldots$,[6], [10] and we consider the class

$$\tilde{L}_{r,q}(X)=\{T\in L(X):\left(\sum_{n=1}^{\infty}\left(n^{\frac{1}{r}}\alpha_n(T)\right)^q\cdot n^{-1}\right)^{\frac{1}{q}}<\infty\},$$

with the quasi-norm

$$\| T \|^{*}_{r,q}=\left(\sum\left(n^{\frac{1}{r}}\alpha_n(T)\right)^q\cdot n^{-1}\right)^{\frac{1}{q}},0<r,q<\infty.$$

Theorem 2.2 *The quasi-norm* $\| T \|^{*}_{r,q}$ *is equivalent with* $\| T \|_{\infty,q,\gamma}$ *if* $\gamma=\frac{1}{r}-\frac{1}{q}$.

Proof. We consider $\log_2 n$ and we obtain

$$\| T \|_{\infty,q,r} = \left(\sum_1^{\infty} [(1+\log n)^{\gamma} s_n(T)]^q n^{-1}\right)^{\frac{1}{q}} =$$

$$= \left(\sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} [(1+\log k)^{\gamma} s_k(T)]^q k^{-1}\right)^{\frac{1}{q}} \leq$$

$$\leq \left(\sum_{n=1}^{\infty} (\max\{(1+\log 2^{n-1})^{\gamma}, (1+\log 2^n)^{\gamma}\} \cdot s_{2^{n-1}}(T))^q \cdot 2^{1-n} 2^{n-1}\right)^{\frac{1}{q}} =$$

$$= \left(\sum_{n=1}^{\infty} (\max\{n^{\gamma}, (n+1)^{\gamma}\} \cdot \alpha_n(T))^q\right)^{\frac{1}{q}} \leq c_1(\gamma, q) \left(\sum_1^{\infty} (n^{\gamma} \alpha_n(T))^q\right)^{\frac{1}{q}} =$$

$$c_1(\gamma, q) \left(\sum_1^{\infty} \left(n^{\frac{1}{r}-\frac{1}{q}} \alpha_n(T)\right)^q\right)^{\frac{1}{q}} = c_1(\gamma, q) \| T \|_{r,q}^{*} .$$

On the other hand

$$\| T \|_{\infty,q,\gamma} \geq \left(s_1^q(T) + \sum_1^{\infty} \left(\min\{(1+\log 2^{n-1})^{\gamma}, (1+\log 2^n)^{\gamma}\} s_{2^n}(T)\right)^q \cdot 2^{-n} \cdot 2^{n-2}\right)^{\frac{1}{q}} =$$

$$= \left(s_1^q(T) + \sum_1^{\infty} (\min\{n^{\gamma}, (n+1)^{\gamma}\} \alpha_{n+1}(T))^q \cdot 2^{-2}\right)^{\frac{1}{q}} =$$

$$\left(\alpha_1^q(T) + \sum_1^{\infty} (\min\{n^{\gamma}, (n+1)^{\gamma}\} \alpha_{n+1}(T))^q \frac{1}{4}\right)^{\frac{1}{q}} \geq$$

$$\geq c(\gamma, q) \| T \|_{r,q}^{*} .$$

Hence

$$c(\gamma, q) \| T \|_{r,q}^{*} \leq \| T \|_{\infty,q,\gamma} \leq c_1(\gamma, q) \| T \|_{r,q}^{*} .$$

Remark. If $\gamma = 0$ it results $r = q$ and

$$\| T \|_{\infty,q} = \left(\sum_1^{\infty} \frac{s_n^q(T)}{n}\right)^{\frac{1}{q}} \sim \| T \|_q^{*} = \left(\sum \alpha_n^q(T)\right)^{\frac{1}{q}}$$

(see [6]).

4. Applications

If X and Y are normed spaces, and $(X, Y)_{\theta,p}$ is the interpolation space, [1], $(0 < p < \infty$, $\theta \in (0, 1)$, it is known [10] that the $\epsilon-$ entropy numbers verify the inequality:

$$\epsilon_{n^2}(T : (X, Y)_{\theta,p} \to A) \leq 4\epsilon_n(T_0 : X \to A)^{1-\theta} \epsilon_n(T_1 : Y \to A)^{\theta},$$

where A is an other normed space $(T = T_0 + T_1.)$

By using the Hölder type inequality it results:

Proposition $L_{\overline{\Phi}_{(p_0)}}(X,A)\cap L_{\overline{\Phi}_{(p_1)}}(Y,A)\subset L_{\overline{\Phi}_{(p)}}((X,Y)_{\theta,p},A)$, for

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\ \ 0<p_0<p_1<\infty,\ \ \epsilon\in(o,\infty).$$

Proof. We recall that $L_{\overline{\Phi}_p}(X,Y)=\{T:\ \overline{\Phi}_{(p)}(\{(\epsilon_n(T))\})\}$, where ϕ is a symmetric norming function and $\overline{\Phi}_{(p)}$ has been defined above.

Now, since $||T||_{\overline{\Phi}_{(p)}}\sim||\overline{T}||_{\overline{\Phi}_{(p)}}$, for $r=2$ (theorem 1.3), we obtain:

$$\overline{\Phi}_{(p)}\left(\{\epsilon_n(T:(X,Y)_{\theta,p}\to A)\right)\sim\overline{\Phi}_p(\{\epsilon_{n^2}(T:(X,Y)_{\theta,p}\to A)\})\leq$$
$$4\overline{\Phi}_{\left(\frac{p_0}{1-\theta}\right)}\left(\epsilon_n(T_0:X\to A)^{1-\theta}\right)\overline{\Phi}_{\left(\frac{p_1}{\theta}\right)}\left(\epsilon_n(T_1:Y-A)^{\theta}\right)=$$
$$4\overline{\Phi}_{(p_0)}\left(\epsilon_n(T_0:X\to A)\right)^{1-\theta}\cdot\overline{\Phi}_{(p_1)}(\epsilon_n(T_1:Y\to A))^{\theta}<\infty.$$

Remark. Recall that the entropy numbers $\epsilon_n(T)$ are defined as follows:

$$\epsilon_n(T)=\inf\{\sigma>0:\exists y_1,y_2,\ldots,y_n\in Y:TU_X\epsilon\cup_{i=1}^{n}\{y_i+\sigma U_Y\}\},$$

where $T:X\to Y$ and $U_X=\{x\in X:\|x\|\leq1\}$.

By means of these numbers it can define the ideals $L_{\overline{\Phi}_{(p)}}(X,Y)$, as above, only for the functions $\overline{\Phi}_{(p)}$, [10], [11], because the numbers $\epsilon_n\ (T)$ are not additive in the classical way.

References

[1] J. Bergh, J. Löfström, *Interpolation spaces*, Springer Verlag, 1976.

[2] I. Gohberg, M. Krein, *Introduction to the theory of non-selfadjoint operators*, A.M.S. Providence, 1969

[3] N. Salinas, *Symmetric norm ideals and relative conjugate ideals*, Trans. A. M.S. 188(1974) 213-240

[4] R. Schatten, *Norm ideals of completely continuous operators*, Springer Verlag 1960

[5] N. Tita, *Operatori de clasă σ_p*, Studii cercet. Mat. 23(1971)467-487

[6] N. Tita, *On a class of $l_{\Phi,\phi}$ operators*, Collect. Mat. 32(1981)275-279

[7] N. Tita, *On Stoltz mappings*, Math. Japonica 26(1981)195-196

[8] N. Tita, *Approximation spaces and biliniar operators*, Studia Univ. Babes-Bolyai 35(1990)89-92

[9] N. Tita, *Some inequalities for the approximation numbers of tensor product operator*, Analele Univ. "Al. I.Cuza"Iasi, 40(1994)329-331

[10] N. Tita, *Ideale de operatori generate de s numere*, Ed. Univ. "Transilvania", Brasov, 1998.

[11] N. Tita, *Cvasinorme echivalente pe spaţii de aproximare*, Ed. Univ. "Transilvania", Braşov, 2001.

[12] N. Tita, *Equivalent quasinorms on some operator ideals*, Ann. Univ. Craiova, 28 (2001) 16 -23.

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