Ученые записки Таврического национального университета им. В. И. Вернадского

Серия "Математика. Механика. Информатика и Кибернетика" N 1 (2003) 198 – 205

УДК 517.4-517.5

ON SUBORDINATED CONDITIONS FOR A SYSTEM OF MINIMAL DIFFERENTIAL OPERATORS IN THE SPACES L_∞

D. V. Limansky
Donetsk National University,
Donetsk, Ukraine

We consider a linear space $L(P_1, \dots, P_N)$ of minimal differential operators subordinated to operators $\{P_j(D)\}_1^N$ in the spaces $L_\infty(\mathbb{R}^n)$. We obtain a criterion for an operator to be quasielliptic in terms of subordinated conditions.

Ключевые слова: quasielliptic differential operator, Bochner theorem, Fourier — Stieltjes transform

1. Introduction

In the paper under consideration we investigate necessary and sufficient conditions for a minimal differential operator $Q(D)$ to be subordinate to a system of other operators $\{P_j(D)\}_1^N$ in the spaces $L_\infty(\mathbb{R}^n)$. In other words, we consider the problem of describing the linear spaces $L(P_1, \dots, P_N)$ of minimal differential operators $Q(D)$ obeying the estimate

$$\|Q(D)f\|_{L_p(\mathbb{R}^n)} \leqslant C\left[\sum\nolimits_{j=1}^N \|P_j(D)f\|_{L_p} + \|f\|_{L_p}\right] \quad \text{for any} \quad f \in C_0^\infty(\mathbb{R}^n) \tag{1}$$

with $p = \infty$ and with some constant C not depending on f. Here $D := (D_1, \dots, D_n)$, $D_j := (1/i)(\partial/\partial x^j)$; $C_0^\infty(\mathbb{R}^n)$ denotes the set of infinitely differentiable functions with compact support.

It is easy to see that for an arbitrary p $(1 \leqslant p \leqslant \infty)$ the condition

$$|Q(\xi)| \leqslant C\left[\sum\nolimits_{j=1}^N |P_j(\xi)| + 1\right], \qquad \xi \in \mathbb{R}^n \tag{2}$$

is necessary for estimate (1) to hold. To prove this, it suffices to substitute in (1) the function $f(x) = g(\varepsilon x) \exp i(x_1\xi_1 + \ldots + x_n\xi_n)$, where $g \in \mathbb{C}_0^\infty(\mathbb{R}^n)$, $g(x) = 1$ in a neighborhood of the origin, and ε is sufficiently small. If $p = 2$, it is easily seen by means of Parseval's formula that (2) is also sufficient for (1) to hold.

V. P. Il'in [5] investigated the case where Q and P_j are monomials and showed that inequality (1) is equivalent to algebraic one (2) for $1 < p < \infty$. J. Boman [3] considered just the same problem for $p = \infty$. He obtained a necessary and sufficient condition of geometric nature for estimate (1) to hold. O. V. Besov [1] and M. M. Malamud [8], [9] (see also [2]) obtained a coercivity criterion, i.e., a criterion for the spaces $L(P_1, \dots, P_N)$ to have maximum possible dimension, for operators with variable coefficients in the cases $1 < p < \infty$ and $p = \infty$ respectively. In the monograph [12], L. R. Volevich and S. G. Gindikin have investigated, in particular, a priori estimates of the type (1) and their applications for finding local smoothness of solutions of partial differential equations. Finally, M. M. Malamud [9] have established type (1) estimates (with $p = \infty$) for systems of minimal differential operators with variable coefficients.

Next, K. De Leeuw and H. Mirkil [6] treated inequality (1) for $p = \infty$, $N = 1$ and $P = P_1$ elliptic. It is well known that if P and Q are partial differential operators with constant coefficients, and P is elliptic, $\deg Q \leqslant \deg P$, then

$$\int |Q(D)f|^2 \leqslant C \int \left(|P(D)f|^2 + |f|^2\right) \quad \text{for any} \quad f \in C_0^\infty. \tag{3}$$

The proof of this "a priori estimate"uses Fourier transforms and the Plancherel theorem. Similar estimates are known for p–th powers ($1 < p < \infty$) in place of squares, although the easy proof for $p = 2$ does not generalize.

In the present paper we investigate the limiting case $p = \infty$ when estimate (1) takes the form

$$||Q(D)f||_{L_\infty(\mathbb{R}^n)} \leqslant C\left[||P(D)f||_\infty + ||f||_\infty\right] \quad \text{for any} \quad f \in C_0^\infty(\mathbb{R}^n). \tag{4}$$

Here $||\cdot||_\infty$ means a standart uniform norm in $\mathbb{R}^n$, i.e.,

$$||u||_{L_\infty(\mathbb{R}^n)} = ||u||_\infty := \sup_{x\in\mathbb{R}^n} |u(x)|.$$

This case turns to be genuinely exceptional. For instance, if $\deg Q = \deg P$ and $Q \neq cP$, then (Proposition 2) an a priori estimate (4) is violated. But if $\deg Q < \deg P$ and P is elliptic, then estimate (4) is reinstated. In fact, in dimension $n \geqslant 3$ this property is characteristic of elliptic operators (Theorem 1), just as the L_2 a priori estimate (3) is characteristic of elliptic operators for the case of equal orders.

The other limiting case $p = 1$ was treated by D. Ornstein [10]. The results for L_1 are essentially the same as those for L_∞. Some counter examples to L_p – estimates of the type (1) were given by M. Littman, C. McCarthy, and N. Rivière [7].

2. Preliminaries and the main theorem

Below, we introduce some important notions and definitions.

Take the set $\mathbb{Z}_+$ of nonnegative integers. If $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $\xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we write

$$\xi^\alpha := \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}, \qquad D^\alpha := D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \qquad D_k = (1/i)(\partial/\partial x^k).$$

In addition, we put $|\alpha : l| := \alpha_1/l_1 + \ldots + \alpha_n/l_n$ whenever $l_1, \ldots, l_n > 0$. $\mathbb{C}[\xi]$ is regarded as a ring of polynomials in n real variables $\xi_1, \ldots, \xi_n$ with complex coefficients. An element from $\mathbb{C}[\xi]$ will be written as $\sum a_\alpha \xi^\alpha$.

We denote by $o(t^k)$ a function that satisfies $\lim_{t\to+\infty} \frac{o(t^k)}{t^k} = 0$ and by $O(t^k)$ a function such that the function $\frac{O(t^k)}{t^k}$ is bounded as $t \to +\infty$ (i.e., in a neighborhood of infinity). Here $k \in \mathbb{Z}_+$.

Definition 1. We say that a differential operator $P(D)$ *dominates* an operator $Q(D)$ (or, equivalently, $Q(D)$ is *subordinated* to $P(D)$) if estimate (4) holds true. Such domination we write as the inclusion $Q \in L(P)$.

Definition 2. Let positive numbers $l_1, \ldots, l_n$ are being fixed. A polynomial $P^l(\xi) \in \mathbb{C}[\xi]$ of the form

$$P^l(\xi) = \sum\nolimits_{|\alpha:l|=1} a_\alpha \xi^\alpha$$

is said to be l – *homogeneous polynomial* with weigh' ; $l_1^{-1}, \ldots, l_n^{-1}$, i.e.,

$$P^l\left(t^{1/l_1}\xi_1, \ldots, t^{1/l_n}\xi_n\right) = tP^l(\xi_1, \ldots, \xi_n), \quad t > 0, \quad \xi \in \mathbb{R}^n.$$

If numbers $l_1, \ldots, l_n$ are natural (i.e., nonzero integer) then we use a notation $[l_1, \ldots, l_n]$ for their least common multiple (LCM).

Definition 3. Differential operator $P(D)$ as well as its symbol $P(\xi) \in \mathbb{C}[\xi]$ is called *quasielliptic* if its principal l – homogeneous part $P^l(\xi)$ vanishes nowhere except at the origin, i.e.,

$$P^l(\xi) = \sum\nolimits_{|\alpha:l|=1} a_\alpha \xi^\alpha = 0 \iff \xi = 0 \quad (\xi \in \mathbb{R}^n).$$

If $l_1 = \ldots = l_n$ then both operator $P(D)$ and its symbol $P(\xi)$ are called *elliptic.*

Let us return to key results from [6] and then impose our main assertions.

Theorem 1. [6] *Let P be a polynomial in $n \geqslant 3$ variables, of degree $d \geqslant 2$. Then a necessary and sufficient condition for P to be elliptic is that P dominates all polynomials of degree $\leqslant d-1$. If $n = 2$, the condition is only sufficient.*

Remark 9. In other words, de Leeuw and Mirkil proved that ellipticity of operator P is sufficient for the inclusion $Q \in L(P)$ to be valid for all Q satisfy $\deg Q < \deg P$. They showed that this condition is also necessary for P to be elliptic operator if $n \geqslant 3$, and is not necessary if $n = 2$. Indeed, the operator $P(D) = P(D_1, D_2) = (D_1 + I)(D_2 + I)$ being nonelliptic dominates operators D_1 and D_2. Here I is the identity operator.

The main aim of the paper is to prove the next theorem which generalizes Theorem 1.

Theorem 2. *Let $l = (l_1, \ldots, l_n)$, where $\{l_i\}_1^n$ are natural numbers not all equal to each other. Let $P(D)$ be differential polynomial in $n \geqslant 3$ variables with principal l – homogeneous part $P^l(D)$. Then the operator $P(D)$ is quasielliptic if and only if $P(D)$ dominates in $L_\infty(\mathbb{R}^n)$ all differential polynomials $Q(D)$ of the form*

$$Q(D) = \sum\nolimits_{|\beta:l|<1} b_\beta D^\beta. \tag{5}$$

Remark 10. The latter condition has simple geometrical meaning. Namely, all points $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ corresponding to degrees of monomials $b_\beta \xi^\beta$ from (5) are situated below the hyperplane $x_1/l_1 + \ldots + x_n/l_n = 1$.

3. Auxiliary statements

In this section we state a number of auxiliary assertions which we use through §4.

Proposition 1. [6] The inclusion $Q \in L(P)$ holds if and only if there exist complex – valued integrable measures μ and ν in $\mathbb{R}^n$ such that

$$Q(\xi) \equiv M(\xi)P(\xi) + N(\xi), \qquad \xi \in \mathbb{R}^n, \tag{6}$$

where $M = \hat{\mu}$ and $N = \hat{\nu}$ are Fourier — Stieltjes transforms defined by

$$M(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi\lambda}\, d\mu(\lambda), \qquad N(\xi) = \hat{\nu}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi\lambda}\, d\nu(\lambda).$$

Proposition 2. [9] Let $l = (l_1, \ldots, l_n)$, $l_i > 0$, $P^l(\xi)$ and $Q^l(\xi)$ be principal l – homogeneous parts of $P(\xi)$ and $Q(\xi)$ respectively. Then the inclusion $Q \in L(P)$ implies the identity

$$Q^l(\xi) \equiv cP^l(\xi), \qquad \xi \in \mathbb{R}^n,$$

with some $c \in \mathbb{C}$.

Remark 11. In [6] Proposition 2 was established only in a homogeneous case $(l_1 = \ldots = l_n)$. There was also demonstrated a connection between Propositions 1 and 2 in the indicated case. Namely, the relation $c = \mu(0)$ takes place, where $\hat{\mu} = M$. Specifically, it follows that $\mu(0) = 0$ if $\deg Q < \deg P$.

Recall [11] that a complex - valued bounded continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be *positive definite function* if the matrix $\|f(\lambda_i - \lambda_j)\|_{1\leqslant i,j\leqslant k}$ is positive for any $k \in \mathbb{N}$ and any $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^k$:

$$\sum_{i,j=1}^{k} f(\lambda_i - \lambda_j)\xi_i\overline{\xi_j} \geqslant 0, \qquad \xi_1, \ldots, \xi_k \in \mathbb{C}, \qquad \lambda_1, \ldots, \lambda_k \in \mathbb{R}^k.$$

Theorem 3. ***(Bochner theorem)*** [11] *Fourier — Stieltjes transforms of all finite positive measures in $\mathbb{R}^n$ form exactly the cone of positive definite functions.*

Lemma 1. *Let μ be a complex - valued finite measure in $\mathbb{R}^n$ and $M = \hat{\mu}$ be its Fourier — Stieltjes transform. Then a restriction of the function M to an arbitrary linear subspace $Z \subset \mathbb{R}^n$ is also a Fourier - Stieltjes transform of some finite measure in Z.*

Доказательство. We can represent the measure μ in the form $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\{\mu_j\}_1^4$ are real - valued positive finite measures. Putting $M_j := \hat{\mu}_j$, $1 \leqslant j \leqslant 4$ we obtain $M = M_1 - M_2 + i(M_3 - M_4)$ and, after restriction to Z, $M|_Z = M_1|_Z - M_2|_Z + i(M_3|_Z - M_4|_Z)$. By Bochner theorem, the functions $\{M_j\}_1^4$ are positive definite, so the functions $\{M_j|_Z\}_1^4$ are positive definite too. By the same theorem, there exist positive finite measures $\{\varkappa_j\}_1^4$ in Z such that $M_j|_Z = \hat{\varkappa}_j$, $1 \leqslant j \leqslant 4$. Finally, we see that the measure $\varkappa := \varkappa_1 - \varkappa_2 + i(\varkappa_3 - \varkappa_4)$ is finite and satisfies $\hat{\varkappa} = M|_Z$. Q. e. d. □

Lemma 2. *Under restriction of l- homogeneous polynomial P^l to a "coordinate subspace"$Z = \{x_{i_1} = \ldots = x_{i_k} = 0,\ 1 \leqslant i_1, \ldots, i_k \leqslant n\}$, $\tilde{P}^l := P^l|_Z$ is also l - homogeneous polynomial.*

Доказательство. Let $\tilde{P}^l \not\equiv 0$ and, for instance, $Z = \{x_n = 0\}$. Then P^l isn't divided by x_n. Hence there exist a nonzero monomial that is a part of P^l and doesn't contain the variable x_n. It follows that this monomial being l - homogeneous comes in $\tilde{P}^l$ too.

For general Z we at first restrict the polynomial P^l to the subspace $\{x_{i_1} = 0\}$, then obtained polynomial is restricted to $\{x_{i_2} = 0\}$, etc. Lemma is proved. □

Corollary 1. *If Z is an arbitrary linear subspace in $\mathbb{R}^n$, then the inclusion $Q \in L(P)$ implies the inclusion $Q|_Z \in L(P|_Z)$. In other words, subordination of one operator to other is reserved under restriction of these operators to an arbitrary subspace.*

Доказательство. A proof is immediately follows from Proposition 1 and Lemmas 1, 2. □

Remark 12. It is easy to show that if Z is a subspace preserving l - homogeneity of polynomial P^l (in particular, this happens if Z is a coordinate subspace) and $\dim Z \geqslant 2$, $\alpha|_Z \neq 0$, then $P^l|_Z \not\equiv 0$. If the above conditions on the subspace Z are fulfilled then all our considerations remain valid in a smaller dimension ($\leqslant n - 1$). This allow us to maintain our argument by induction on dimension n throughout §4.

Lemma 3. ***(Eberlein lemma)*** [4] *Let μ be an integrable measure, let c be the signed mass of μ of the origin, and let $M = \hat{\mu}$. Then the constant function c can be approximated uniformly by $\pi * M$, with π a probability measure.*

Corollary 2. *If μ is an integrable measure in $\mathbb{R}^n$ and $\mu(0) = 0$ then zero can be approximated uniformly by convex combinations of translates of the function $M = \hat{\mu}$, i.e.,*

$$\sum_{k=1}^{m} c_k M(\xi - \xi_k) \rightrightarrows 0, \qquad c_k > 0, \qquad \sum_{1}^{m} c_k = 1, \qquad \xi,\ \xi_k \in \mathbb{R}^n.$$

In the sequel, we assume that a polynomial Q is choosen and fixed, and $\deg Q < \deg P$. So functions $M = \mu$ and $N = \nu$ are also fixed in (6) and, by Remark 11, $\mu(0) = 0$.

Definition 4. Consider a family Γ of *"polynomial curves"* in $\mathbb{R}^n$ defined parametrically:

$$x = x(t) := (x_1(t), \dots, x_n(t)), \quad x_i = x_i(t) \in \mathbb{R}[t], \quad t \geqslant 0.$$

We say that Γ is *an admissible family* if:

1) there is a subfamily $\tilde{\Gamma} \subseteq \Gamma$ such that $\forall \theta \in \mathbb{R}^n$, $\forall x \in \tilde{\Gamma} \Longrightarrow x - \theta \in \tilde{\Gamma}$;
2) $\lim_{t\to+\infty} M(x(t)) = 1$ whenever $x = x(t) \in \tilde{\Gamma}$.

Proposition 3. Under above assumptions, there are no admissible families of polynomial curves in $\mathbb{R}^n$.

Доказательство. Suppose Γ is the admissible family. Eberlein lemma yields

$$\exists \theta_1, \dots, \theta_k \in \mathbb{R}, \quad \exists x_1, \dots, x_k \in \mathbb{R}^n : \quad \left|\sum_{i=1}^{k} \theta_i M(x - x_i)\right| < 1/2 \quad \text{for any} \quad x \in \mathbb{R}^n. \tag{7}$$

Let us substitute an arbitrary polynomial curve $x = x(t) \in \tilde{\Gamma} \subseteq \Gamma$ for x in (7). Then $y_i(t) := x(t) - x_i \in \tilde{\Gamma}$ and $\lim_{t\to\infty} M(y_i(t)) = 1$, $1 \leqslant i \leqslant k$. Therefore,

$$\left|\sum_{i=1}^{k} \theta_i M(y_i(t))\right| < 1/2, \quad t \geqslant 0. \tag{8}$$

Turning $t \to \infty$ in both sides of inequality (8), we obtain apparently wrong relation $1 = |\sum_1^k \theta_i| < 1/2$. The contradiction completes the proof. □

We also need the following number-theoretic lemma.

Lemma 4. *Let $l = (l_1, \dots, l_n) \in \mathbb{N}^n$, $n \geqslant 2$, and let $[l_1, \dots, l_n] > l_i$ $\forall i \in [1; n]$. Consider the set I of all $k \in \mathbb{Z}_+^n$ for which $|k : l| = \sum_1^n k_i/l_i < 1$. Then $\max_{k\in I} |k : l| > 1 - (1/\max_{1\leqslant i\leqslant n} l_i)$.*

4. Outline of the proof of Theorem 2

M. M. Malamud [9] has already proved the sufficiency of the statement (i.e., that quasiellipticity of P yields the subordination of all Q of the form (5) to P). We prove the necessity.

Suppose the converse, i.e., that $Q \in L(P)$ for all Q of the form (5) but P is not quasielliptic. It follows that $\exists \alpha \in \mathbb{R}^n$, $\alpha \neq 0$: $P^l(\alpha) = 0$. Without loss of generality, we can assume that $l_1 \geqslant l_2 \geqslant \dots \geqslant l_n$. Denote by $d := [l_1, \dots, l_n]$ the LCM of $l_1, \dots, l_n$ and put $m_i := d/l_i$, so that $m_1 \leqslant m_2 \leqslant \dots \leqslant m_n$. We write $P(\xi)$ as the decomposition $P(\xi) = P^d(\xi) + P^{d_1}(\xi) + \dots$, where P^{d_i} are "quasihomogeneous forms of quasidegree d_i". Namely, we mean that each $P^{d_i}(\xi)$, $d_0 := d$, $d_1 > d_2 > \dots$ is the sum of monomials $a_k\xi_1^{k_1} \dots \xi_n^{k_n}$ involved in $P(\xi)$ such that $\sum_{j=1}^n m_j k_j = d_i$. In particular, $P^d(\xi) \equiv P^l(\xi)$.

Using identity (6), it is easy to show that $P^{d_1}(\alpha) \neq 0$. So we normalize the polynomial P by setting $P^{d_1}(\alpha) = 1$. Let also

$$u_i := \frac{\partial P^d}{\partial x^i}(\alpha), \qquad 1 \leqslant i \leqslant n. \tag{9}$$

Stress that u_i are, in general, complex numbers. Choose $Q(\xi) := P^{d_1}(\xi)$ in (6), so that

$$P^{d_1}(\xi) \equiv M(\xi)P(\xi) + N(\xi), \qquad \xi \in \mathbb{R}^n. \tag{10}$$

4.1. **Case $d > l_i$.** First we consider more simpler case, where $d > l_i$, $i \in \{1, \dots, n\}$.

Consider the following functions (that define some family $\mathcal{N}$ of polynomial curves):

$$x_i(t) := \alpha_i t^{m_i} + a_i, \qquad 1 \leqslant i \leqslant n, \tag{11}$$

with arbitrary $a_i \in \mathbb{R}$. Substituting functions (11) for ξ in (10) and expanding each polynomial P^{d_i} according to Taylor's formula, we have

$$t^{d_1} + o(t^{d_1}) = M(x(t)) \left[\sum_{i=1}^{n} t^{d-m_i} u_i a_i + t^{d_1} + o(t^{d_1})\right] + N(x(t)), \quad \text{as} \quad t \to +\infty.$$

We note that $d_1 = \max_{k\in I} |k : l|$ (in notations of Lemma (4)) and, by the same lemma, $d_1 > d - m_i$, $i \in \{1, \dots, n\}$. Thus $P^d(\alpha_1 t^{m_1}, \dots, \alpha_n t^{m_n}) = o(t^{d_1})$ and

$$t^{d_1} + o(t^{d_1}) = M(x(t))\left[t^{d_1} + o(t^{d_1})\right] + N(x(t)), \quad \text{as} \quad t \to +\infty. \tag{12}$$

Dividing both members of (12) by t^{d_1} and turning $t \to \infty$ we obtain $\lim_{t\to+\infty} M(x(t)) = 1$ for any $x(t) \in \mathcal{N}$. Note that functions (11) satisfy all requirements of Definition 4 and hence the family $\mathcal{N}$ defined by these functions is admissible. This contradicts Proposition 3. Theorem 2 is proved if $d > l_i$.

4.2. **Case** $d = l_1$. Now let $d = l_1 = \max_{1\leqslant i\leqslant n} l_i$. Then $d_i = i$, $i \in \{1, \dots, n\}$, $m_1 = 1$, $P^{d_1}(\alpha) = P^{d-1}(\alpha) = 1$.

Consider the family $\mathcal{P}$ of polynomial curves defined by

$$x_1(t) := \alpha_1 t + a_1; \quad x_i(t) = \alpha_i t^{m_i} + a_i t^{m_i - 1} + \dots, \qquad 2 \leqslant i \leqslant n, \tag{13}$$

with at first arbitrary $a_i \in \mathbb{R}$. Replacing ξ by functions (13) in (10) and expanding each polynomial P^{d_i} according to Taylor's formula, after not complicated computations we obtain

$$t^{d-1} + o(t^{d-1}) = M(x(t))\left[\left(\sum\nolimits_{i=1}^{n} u_i a_i + 1\right) t^{d-1} + o(t^{d-1})\right] + N(x(t)), \quad \text{as} \quad t \to +\infty. \tag{14}$$

Dividing both members of (14) by t^{d-1} then turning $t \to \infty$ and taking into account that $M = O(1)$ and $N = O(1)$, we have

$$\lim_{t\to+\infty}\left[M(x(t))\left(\sum\nolimits_{i=1}^{n} u_i a_i + 1\right)\right] = 1. \tag{15}$$

Two situations concerning numbers $\{u_i\}_1^n$ (9) are possible.

1. Vectors $\Re u := (\Re u_1, \dots, \Re u_n)$ and $\Im u := (\Im u_1, \dots, \Im u_n)$ are not proportional. Then there exist real numbers $\{a_i\}_1^n$ such that $\sum_1^n u_i a_i + 1 = 0$. This contradicts equality (15) and completes the proof of the theorem.

2. Vectors $\Re u$ and $\Im u$ are proportional (including the case when one of them equals to 0). Restrict a choice of numbers $\{a_i\}_1^n$ by the equality

$$\sum\nolimits_{i=1}^{n} u_i a_i = 0. \tag{16}$$

Then (16) implies that $\lim_{t\to\infty} M(x(t)) = 1$, with $x(t)$ satisfy (13).

Let us partition the numbers $\{m_i\}_1^n$ into p groups such that all numbers within each group are equal, i.e.,

$$1 = m_1 = \dots = m_{j_1} < m_{j_1+1} = \dots = m_{j_2} < \dots < m_{j_{p-1}+1} = \dots = m_{j_p} = m_n,$$
$$1 \leqslant j_1 < j_2 < \dots < j_p = n.$$

It implies the partition of the sum $\sum_1^n u_i \alpha_i$ into p sums

$$v_k := \sum\nolimits_{i=j_{k-1}+1}^{j_k} u_i \alpha_i, \qquad j_0 := 0, \qquad 1 \leqslant k \leqslant p.$$

Next, we shift the parameter t in (13), i.e., we replace t in functions $x(t)$ (13) by $(t+\tau)^k$, $k \in \{1, \dots, p\}$, where τ is arbitrary constant. Such shifts give us another functions $\tilde{x}(t)$ that define just the same geometric objects (curves) as $x(t)$ do. Hence, $\lim_{t\to\infty} M(\tilde{x}(t)) = 1$. Combining this with equalities (10), (16) we obtain after some computations that the vector $v := (v_1, \dots, v_p)$ is a solution of a homogeneous system of p linear equations with a Vandermonde type determinant. It follows that

$$v_k = 0 \quad \text{for any} \quad k \in \{1, \dots, p\}. \tag{17}$$

Now we can easily complete the proof by induction on dimension n. First we analyze the case $n = 3$ which is the induction base, and see for oneself that Theorem 2 is true (we omit these considerations due to the shortage of space in this paper). Secondly we suppose that $n > 3$ and the theorem holds true for all dimensions k, $3 \leqslant k \leqslant n-1$, and consider three main cases concerning the vector α.

1) Let $\exists j \in \{1, \dots, n\}$: $\alpha_j = 0$. Therefore we can restrict our problem to the subspace $Z := \{x_j = 0\}$, $\dim Z \leqslant n-1$. Then $P^l|_Z(\alpha_1, \dots, \alpha_{n-1}) \neq 0$ and, by Corollary 1 and further Remark 12, $P^l|_Z$ dominates (in the dimension $n-1$) all polynomials Q of the form (5). This contradicts the induction hypothesis.

2) Let $\alpha_i \neq 0$, $i \in \{1, \dots, n\}$ and $1 = m_1 < \dots < m_n$. Then $0 = v_k = u_k \alpha_k$, $k \in \{1, \dots, p\}$, $p = n$ and, hence, $u_i = 0$, $i \in \{1, \dots, n\}$. This means that a choice of parameters $\{a_i\}_1^n$ in (13) is unrestricted and the family $\mathcal{P}$ is admissible.

3) Finally, let $\alpha_i \neq 0$, $i \in \{1, \dots, n\}$ and $\exists w_1 \neq w_2$: $m_{w_1} = m_{w_2}$. It follows that our problem becomes homogeneous within the subspace $Z := \{x_{w_1} = x_{w_2} = 0\}$ (we mean that $l_{w_1} = l_{w_2}$). Thus by suitable linear change of variables in Z (for instance, by rotation about the origin) we can achieve that, for example, $\alpha_{w_2} = 0$. So we can pass to the case considered above (item 1). Theorem 2 is completely proved.

5. Short proof of Theorem 1

In this section we reprove Theorem 1 from [6] by means of techniques developed above.

We consider a homogeneous case:

$$l_1 = \dots = l_n = d, \qquad m_1 = \dots = m_n = 1.$$

As above (§4), we are to prove only necessity. Denote by P^k, $0 \leqslant k \leqslant d$, the homogeneous form of degree k involved in P, so that $P = P^d + P^{d-1} + \dots$. Suppose the polynomial P of degree d dominates all polynomials Q of degree $\leqslant d-1$ in $L_\infty(\mathbb{R}^n)$, $n \geqslant 3$ but P is not elliptic. At that time $\exists \alpha \in \mathbb{R}^n$: $P^d(\alpha) = 0$, and $P^{d-1}(\alpha) \neq 0$ (we put $P^{d-1}(\alpha) = 1$). Let u, u_i, $\Re u$, $\Im u$, p and v_k be the same as earlier. Consider the family $\mathcal{P}$ with $m_i = 1$, $i \in \{1, \dots, n\}$, i.e., the family Q of parallel lines in $\mathbb{R}^n$:

$$x_i(t) := \alpha_i t + a_i, \qquad 1 \leqslant i \leqslant n. \tag{18}$$

Substituting functions (18) in the identity

$$P^{d-1} = MP + N = M(P^d + P^{d-1} + \dots) + N$$

and arguing as in §4, subsection 4.2, we pass to the equality (15). It gives a contradiction if the vectors $\Re u$ and $\Im u$ are not proportional. So let us assume that $\Re u$ and $\Im u$ are proportional. Restrict a choice of $\{a_i\}_1^n$ by equality (16). Besides (see (17)),

$$p = 1, \qquad v_1 = \sum_{i=1}^{n} u_i \alpha_i = 0. \tag{19}$$

It follows from (19), (16) that functions (18) satisfy the linear equation

$$u_1 x_1(t) + \dots + u_n x_n(t) = 0.$$

If $u_1 = \dots = u_n = 0$, there are no restrictions to $\{a_i\}_1^n$ and consequently the family Q is admissible. At last, if $\exists i \in \{1, \dots, n\}$: $u_i = 0$, then all parallel lines from Q lie on the hyperplane h given by the equation

$$(\Re u_1) x_1 + \dots + (\Re u_n) x_n = 0.$$

We see that the family Q satisfy the requirements of Definition 4 (within h) and hence it is admissible in dimension $n-1$. Theorem 1 is proved.

Acknowledgements. The author wishes to express the gratitude to his supervisor M. M. Malamud for constant attention to the work and useful discussions.

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Limansky D. V., Universitetskaya 24, Math. Dept., Donetsk National University, Donetsk, 83055, Ukraine
*E-mail:*lim@univ.donetsk.ua