Ученые записки Таврического национального университета им. В. И. Вернадского

Серия "Математика. Механика. Информатика и Кибернетика" N 1 (2003) 182 - 185

УДК 517.95

ON MOMENTS OF A SYSTEM OF TWO DIFFERENTIAL EQUATIONS WITH STOCHASTIC PERTURBATIONS.

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The investigation of differential equations as well as systems of differential equations whose coefficients are perturbed by stochstic processes, is of a permanent interes. Such systems describe varions practical problems [1].

We consider the system of two first order differential equations

$$\begin{cases} \frac{dy_1}{dt} &= (a_{11}(t)+\xi_{11}(t,\omega))y_1(t,\omega)+(a_{12}(t)+\xi_{12}(t,\omega))y_2(t,\omega), \\ \frac{dy_2}{dt} &= (a_{21}(t)+\xi_{21}(t,\omega))y_1(t,\omega)+(a_{22}(t)+\xi_{22}(t,\omega))y_2(t,\omega) \end{cases} \quad (1)$$

and the initial conditions

$$y_1(0,\omega)=y_1^0(\omega), \quad y_2(0,\omega)=y_2^0(\omega). \quad (2)$$

Here the stochastic processes $y_1(t,\omega)$, $y_2(t,\omega)$ satisfy the system (1) for allmost all $\omega\in\Omega$ (Ω is a probality space), $y_1^0(\omega)$, $y_2^0(\omega)$ are random variables; $a_{kj}(t)$ $(k,j=1,2)$ are continuons functions on $[0,T]$; $\xi_{kj}(t,\omega)$ are independent non-Gaussian delta-correlated stochastic processes with the means $<\xi_{kj}(t,\cdot)>$ equal to zero. The comulante functions of the processes $\xi_{kj}(t,\omega)$ have the form

$$K_m^{kj}(t_1,\ldots,t_m)=s_m^{kj}(t_1)\delta(t_1-t_2)\cdots\delta(t_{m-1}-t_m) \quad (k,j=1,2),$$

where $\delta(t_s-t_{s+1})$ $(s=1,2,...,m-1)$ are delta-functions.

Let $s_m^{kj}(t)$ are continuons functions, and $\sum\limits_{m=1}^{\infty} m(m+1)\ s_{m+1}^{kj}$ are uniformly converging series on $[0,T]$.

The solution of problem (1)-(2) is functionals of the processes $\xi_{11}(t,\omega)$, $\xi_{12}(t,\omega)$, $\xi_{21}(t,\omega),\xi_{22}(t,\omega):$ $y_k(t,\omega)=F[\xi_{11}(t,\omega),\ \xi_{12}(t,\omega),\ \xi_{21}(t,\omega),\ \xi_{22}(t,\omega)]$ $(k=1,2)$.

In practice it is offen enough to know two of the first moments of the solution of the system: the expectation and the covariance function.

In [2] we obtained the system of ordinary differential equations for determining means of the solution $<y_1(t,\cdot)>$ and $<y_2(t,\cdot)>$. We find the covariance functions $q_{kj}(t,\tau)=\langle y_k(t,\cdot)y_j(\tau,\cdot)\rangle$ $(k,j=1,2)$ of the solution. Denote by $\hat{q}_{kj}(t,\tau)$ the covariance function of the solution for $t>\tau$, and by $\check{q}_{kj}(t,\tau)$ the covariance function of the solution for $t<\tau$.

Let us find the system for determining $\hat{q}_{kj}(t,\tau)$. For this end multiply the equations in (1) by $y_1(\tau,\omega)$ and $y_2(\tau,\omega)$ sequentially, and then average. We obtain

$$\begin{cases} \frac{d\hat{q}_{11}}{dt} &= a_{11}(t)\hat{q}_{11}(t,\tau)+a_{12}(t)\hat{q}_{21}(t,\tau)+ \\ &+ \langle\xi_{11}(t,\cdot)y_1(t,\cdot)y_1(\tau,\cdot)\rangle+\langle\xi_{12}(t,\cdot)y_2(t,\cdot)y_1(\tau,\cdot)\rangle, \\ \frac{d\hat{q}_{21}}{dt} &= a_{21}(t)\hat{q}_{11}(t,\tau)+a_{22}(t)\hat{q}_{21}(t,\tau)+ \\ &+ \langle\xi_{21}(t,\cdot)y_1(t,\cdot)y_1(\tau,\cdot)\rangle+\langle\xi_{22}(t,\cdot)y_2(t,\cdot)y_1(\tau,\cdot)\rangle, \\ \frac{d\hat{q}_{12}}{dt} &= a_{11}(t)\hat{q}_{12}(t,\tau)+a_{12}(t)\hat{q}_{22}(t,\tau)+ \\ &+ \langle\xi_{11}(t,\cdot)y_1(t,\cdot)y_2(\tau,\cdot)\rangle+\langle\xi_{12}(t,\cdot)y_2(t,\cdot)y_2(\tau,\cdot)\rangle, \\ \frac{d\hat{q}_{22}}{dt} &= a_{21}(t)\hat{q}_{12}(t,\tau)+a_{22}(t)\hat{q}_{22}(t,\tau)+ \\ &+ \langle\xi_{21}(t,\cdot)y_1(t,\cdot)y_2(\tau,\cdot)\rangle+\langle\xi_{22}(t,\cdot)y_2(t,\cdot)y_2(\tau,\cdot)\rangle. \end{cases} \quad (3)$$

This system is not closed. To "decompose"the correlation of the stochastic processes and the solution, we find the means $< \xi_{kj}(t,\cdot)y_j(t,\cdot)y_i(\tau,\cdot) >$ $(k,j,i = 1,2)$. We use the formula [3], which in this situation has the form

$$< \xi_{kj}(t,\cdot)y_j(t,\cdot)y_i(\tau,\cdot) >= \sum_{m=1}^{\infty} \frac{s_{m+1}^{kj}(t)}{(m-1)!} \left\langle \frac{\delta^m (y_j(t,\cdot)y_i(\tau,\cdot))}{\delta\xi_{kj}(t,\cdot)\cdots\delta\xi_{kj}(t,\cdot)} \right\rangle, \tag{4}$$

where

$$\frac{\delta^m (y_j(t,\omega)y_i(\tau,\omega))}{\delta\xi_{kj}(t,\omega)\cdots\delta\xi_{kj}(t,\omega)} \quad \text{is } m\text{variational derivative} \quad \frac{\delta^m (y_j(t,\omega)y_i(t,\omega))}{\delta\zeta_{kj}^1(t_1,\omega)\cdots\delta\zeta_{kj}^m(t_m,\omega)}$$

by $\zeta_{kj}^1(t_1,\omega),\ldots,\zeta_{kj}^m(t_m,\omega)$ at the points $t_1,\ldots,t_m$ for $\zeta_{kj}^1 = \cdots = \zeta_{kj}^m = \xi_{kj}$ and for $t_1 = \cdots = t_m = t$.

Substituting (4) into (3), we obtain:

$$\left\{ \begin{array}{rcl} \frac{d\hat{q}_{11}}{dt} & = & a_{11}(t)\hat{q}_{11}(t,\tau) + a_{12}(t)\hat{q}_{21}(t,\tau) + \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{11}}{(m-1)!} \langle \frac{\delta^m (y_1(t,\cdot)y_1(\tau,\cdot))}{\delta\xi_{11}(t,\cdot)\cdots\xi_{11}(t,\cdot)} \rangle + \\ & + & \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{12}}{(m-1)!} \langle \frac{\delta^m (y_2(t,\cdot)y_1(\tau,\cdot))}{\delta\xi_{12}(t,\cdot)\cdots\xi_{12}(t,\cdot)} \rangle, \\ \frac{d\hat{q}_{21}}{dt} & = & a_{21}(t)\hat{q}_{11}(t,\tau) + a_{22}(t)\hat{q}_{21}(t,\tau) + \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{21}}{(m-1)!} \langle \frac{\delta^m (y_1(t,\cdot)y_1(\tau,\cdot))}{\delta\xi_{21}(t,\cdot)\cdots\xi_{21}(t,\cdot)} \rangle + \\ & + & \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{22}}{(m-1)!} \langle \frac{\delta^m (y_2(t,\cdot)y_1(\tau,\cdot))}{\delta\xi_{22}(t,\cdot)\cdots\xi_{22}(t,\cdot)} \rangle, \\ \frac{d\hat{q}_{12}}{dt} & = & a_{11}(t)\hat{q}_{12}(t,\tau) + a_{12}(t)\hat{q}_{22}(t,\tau) + \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{11}}{(m-1)!} \langle \frac{\delta^m (y_1(t,\cdot)y_2(\tau,\cdot))}{\delta\xi_{11}(t,\cdot)\cdots\xi_{11}(t,\cdot)} \rangle + \\ & + & \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{12}}{(m-1)!} \langle \frac{\delta^m (y_2(t,\cdot)y_2(\tau,\cdot))}{\delta\xi_{12}(t,\cdot)\cdots\xi_{12}(t,\cdot)} \rangle, \\ \frac{d\hat{q}_{22}}{dt} & = & a_{21}(t)\hat{q}_{12}(t,\tau) + a_{22}(t)\hat{q}_{22}(t,\tau) + \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{21}}{(m-1)!} \langle \frac{\delta^m (y_1(t,\cdot)y_2(\tau,\cdot))}{\delta\xi_{21}(t,\cdot)\cdots\xi_{21}(t,\cdot)} \rangle + \\ & + & \sum\limits_{m=1}^{\infty} \frac{s_{m+1}^{22}}{(m-1)!} \langle \frac{\delta^m (y_2(t,\cdot)y_2(\tau,\cdot))}{\delta\xi_{22}(t,\cdot)\cdots\xi_{22}(t,\cdot)} \rangle. \end{array} \right. \tag{5}$$

Using the causality principle, it can be determined the variational derivatives

$$\frac{\delta^m (y_j(t,\omega)y_i(\tau,\omega))}{\delta\xi_{kj}(t,\omega)\cdots\delta\xi_{kj}(t,\omega)}$$

in the same way as we have done in [2]. We have

$$\frac{\delta^m (y_j(t,\omega)y_i(\tau,\omega))}{\delta\xi_{kj}(t,\omega)\cdots\delta\xi_{kj}(t,\omega)} = m!y_k(t,\omega)y_i(\tau,\omega) \quad \text{for} \quad k = j, \tag{6}$$

$$\frac{\delta^m (y_j(t,\omega)y_i(\tau,\omega))}{\delta\xi_{kj}(t,\omega)\cdots\delta\xi_{kj}(t,\omega)} = 0 \quad \text{for} \quad k \neq j \qquad (i = 1,2), \tag{7}$$

$$\frac{\delta^m (y_k(t,\omega)y_k(t,\omega))}{\delta\xi_{kk}(t,\omega)\cdots\delta\xi_{kk}(t,\omega)} = (m+1)!\, y_k^2(t,\omega) \quad (k,j = 1,2). \tag{8}$$

Taking into account (6), (7), we obtain from system (5) the closed system of four ordinary differential equations. It is represented by two systems every of which consists of two equations:

$$\begin{cases} \frac{d\hat{q}_{11}}{dt} &= \Big(a_{11}(t) + \sum\limits_{m=1}^{\infty} m\ s^{11}_{m+1}(t)\Big)\hat{q}_{11}(t,\tau) + a_{12}(t)\hat{q}_{21}(t,\tau), \\ \frac{d\hat{q}_{21}}{dt} &= a_{21}(t)\hat{q}_{11}(t,\tau) + \Big(a_{22}(t) + \sum\limits_{m=1}^{\infty} m\ s^{22}_{m+1}(t)\Big)\hat{q}_{21}(t,\tau), \\ \frac{d\hat{q}_{12}}{dt} &= \Big(a_{11}(t) + \sum\limits_{m=1}^{\infty} m\ s^{11}_{m+1}(t)\Big)\hat{q}_{12}(t,\tau) + a_{12}(t)\hat{q}_{22}(t,\tau), \\ \frac{d\hat{q}_{22}}{dt} &= a_{21}(t)\hat{q}_{12}(t,\tau) + \Big(a_{22}(t) + \sum\limits_{m=1}^{\infty} m\ s^{22}_{m+1}(t)\Big)\hat{q}_{22}(t,\tau). \end{cases} \tag{9}$$

The initial condidions for system (9) are dispersions, because $\hat{q}_{kj}(t,t) = \check{q}_{kj}(t,t) = q_{kj}(t,t) = D_{kj}(t)$ $(k,j = 1,2,\ D_{kj}(t) = D_{jk}(t))$ at the point $t = \tau$.

So, we obtain the system

$$\begin{cases} \frac{d\hat{q}_{11}}{dt} &= \Big(a_{11}(t) + \sum\limits_{m=1}^{\infty} m\ s^{11}_{m+1}(t)\Big)\hat{q}_{11}(t,\tau) + a_{12}(t)\hat{q}_{21}(t,\tau), \\ \frac{d\hat{q}_{21}}{dt} &= a_{21}(t)\hat{q}_{11}(t,\tau) + \Big(a_{22}(t) + \sum\limits_{m=1}^{\infty} m\ s^{22}_{m+1}(t)\Big)\hat{q}_{21}(t,\tau) \end{cases} \tag{10}$$

with the initial conditions

$$\hat{q}_{11}(t,t) = D_{11}(t), \quad \hat{q}_{21}(t,t) = D_{12}(t), \tag{11}$$

and the system

$$\begin{cases} \frac{d\hat{q}_{12}}{dt} &= \Big(a_{11}(t) + \sum\limits_{m=1}^{\infty} m\ s^{11}_{m+1}(t)\Big)\hat{q}_{12}(t,\tau) + a_{12}(t)\hat{q}_{22}(t,\tau), \\ \frac{d\hat{q}_{22}}{dt} &= a_{21}(t)\hat{q}_{12}(t,\tau) + \Big(a_{22}(t) + \sum\limits_{m=1}^{\infty} m\ s^{22}_{m+1}(t)\Big)\hat{q}_{22}(t,\tau) \end{cases} \tag{12}$$

with the initial conditions

$$\hat{q}_{12}(t,t) = D_{12}(t), \quad \hat{q}_{22}(t,t) = D_{22}(t). \tag{13}$$

Analogously we can get the system for $t < \tau$, hence for $\check{q}_{kj}(t,\tau)$.

To find dispersions, we have the system of three differential equations of the form

$$\frac{dD_{11}}{dt} = 2\left[\left(a_{11}(t) + \sum_{m=1}^{\infty} m(m+1)\ s^{11}_{m+1}(t)\right) D_{11}(t) + a_{12}(t)D_{12}(t)\right],$$

$$\frac{dD_{12}}{dt} = a_{21}(t)D_{11}(t) + a_{12}(t)D_{22}(t) + \left[a_{11}(t) + a_{22}(t) + \sum_{m=1}^{\infty} m(m+1)\ \left(s^{11}_{m+1}(t) + s^{22}_{m+1}(t)\right)\right] D_{12}(t$$

$$\frac{dD_{22}}{dt} = 2\left[a_{21}(t)D_{12}(t) + \left(a_{22}(t) + \sum_{m=1}^{\infty} m(m+1)\ s^{22}_{m+1}(t)\right) D_{22}(t)\right]$$

with the initial conditions

$$D_{kj}(0) = < y^0_k(\cdot) y^0_j(\cdot) > .$$

If $\xi_{k,j}(t,\omega)$ $(k,j = 1,2)$ are Gaussian delta-correlated processes, the system (9) is simplified, because $K^{kj}_1(t) = < \xi_{kj}(t) > = 0$ (by conditions), $K^{kj}_2(t_1,t_2) =$ $< \xi_{kj}(t_1,\cdot)\xi_{kj}(t_2,\cdot) > = s^{kj}_2(t_1)\delta(t_1 - t_2)$, and all the other cumulants $K^{kj}_m(t_1,t_2,\ldots,t_m)$ $(m = 3,4,\ldots)$ are equal to zero. Thus, the systems (10) and (12) take the forms:

$$\begin{cases} \frac{d\hat{q}_{11}}{dt} &= \Big(a_{11}(t) + s^{11}_2(t)\Big)\hat{q}_{11}(t,\tau) + a_{12}(t)\hat{q}_{21}(t,\tau), \\ \frac{d\hat{q}_{21}}{dt} &= a_{21}(t)\hat{q}_{11}(t,\tau) + \Big(a_{22}(t) + s^{21}_2(t)\Big)\hat{q}_{21}(t,\tau) \end{cases}$$

with the initial conditions (11)

and

$$\begin{cases} \frac{d\hat{q}_{12}}{dt} &= \Big(a_{11}(t) + s_2^{12}(t)\Big)\hat{q}_{12}(t,\tau) + a_{12}(t)\hat{q}_{22}(t,\tau), \\ \frac{d\hat{q}_{22}}{dt} &= a_{21}(t)\hat{q}_{12}(t,\tau) + \Big(a_{22}(t) + s_2^{22}(t)\Big)\hat{q}_{22}(t,\tau) \end{cases}$$

with the initial conditions (13).

The system for determiningdispersions in this case is of the form

$$\frac{dD_{11}}{dt} = 2\left[\left(a_{11}(t) + s_2^{11}(t)\right) D_{11}(t) + a_{12}(t)D_{12}(t)\right],$$

$$\frac{dD_{12}}{dt} = a_{21}(t)D_{11}(t) + a_{12}(t)D_{22}(t) + \left[a_{11}(t) + a_{22}(t) + \left(s_2^{11}(t) + s_2^{22}(t)\right)\right] D_{12}(t),$$

$$\frac{dD_{22}}{dt} = 2\left[a_{21}(t)D_{12}(t) + \left(a_{22}(t) + s_2^{22}(t)\right) D_{22}(t)\right]$$

with the initial conditions

$$D_{kj}(0) = < y_k^0(\cdot)y_j^0(\cdot) > .$$

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Kovtun I.I.

ON MOMENTS OF A SYSTEM OF TWO DIFFERENTIAL EQUATIONS WITH STOCHASTIC PERTURBATIONS. II

We consider a system of two ordinary differential equations with stochastic delta-correlations non-Gaussian perturbations. We obtained the closed system of four ordinary differential equations for determining the covariance function of the solution.

Ковтун И.И.

МОМЕНТЫ СИСТЕМЫ ДВУХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ СО СЛУЧАЙНЫМИ ВОЗМУЩЕНИЯМИ. II

Рассматривается система двух линейных дифференциальных уравнений, возмущенных негауссовскими дельта-коррелированными случайными процессами. Получена замкнутая система четырех обыкновенных дифференциальных уравнений для нахождения ковариационной функции решения.

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