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ON THE REFLEXIVITY OF THE OPERATOR $J_k^\alpha \oplus J_{k+s}^\alpha$

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Introduction. Let X_1, X_2 are Banach spaces. We denote by $[X_1, X_2]$ the space of bounded linear operators from X_1 to X_2, $[X] := [X, X]$. $AlgT$ denotes the weakly closed subalgebra of $[X]$ generated by T and the identity $\mathbb{I}$. $LatT$ denotes the lattice of subspaces of X invariant under T and $AlgLatT$ denotes the algebra of operators in $[X]$ leaving each subspace in $LatT$ invariant.

P.R. Halmos has called a subalgebra $\mathfrak{A}$ of $[X]$ reflexive if the only operators that leave invariant all the closed invariant subspaces of $\mathfrak{A}$ are the members of $\mathfrak{A}$ itself. Reflexive algebras contain the identity operator $\mathbb{I}$ and are closed in the weak operator topology; on the other hand, it follows from the double commutant theorem that any von Neuman algebra (i.e. weakly closed self-adjoint operator algebra containing $\mathbb{I}$) is reflexive. The question of reflexivity of an operator is the question of reflexivity of the weakly closed algebra generated by it. The reflexivity property of an operator has an approximation character: an operator A is reflexive if every operator whose A-invariant subspaces are all invariant can be approximated by polinomials in A in the weak operator topology, i.e. if $AlgLatA = AlgA$. The first results about reflexive operators are contained in D. Sarason's paper [8], were the reflexivity of normal operators and analytic Teoplitz operators is proved. J. A. Deddens [2] has proved that any isometric operator is reflexive. J.A. Deddens and P.A. Fillmore [3] have found a criterion of reflexivity for operators acting on finite - dimensional spaces (see also [4]). For $T \in [X]$ we write $T^{(k)}$ for the direct sum of k-copies of T. The operator T is called k-reflexive if $T^{(k)}$ is reflexive. k-reflexivity of the finite dimensional algebras were investigated by E.Azoff [1]. A reflexivity criterion for contractions with a scalar inner characteristic function has been established by V.V. Kapustin in [6]. He has proved also that a C_0-operator with a Jordan model $M_{\Theta_1} \oplus M_{\Theta_2} \oplus \dots$ is reflexive if and only if M_Θ is reflexive, where $\Theta = \Theta_1/\Theta_2$. In particular, each C_0-contraction is 2-reflexive.

The following fact is implicitly contained in [8] (see also [7] for the explicit statement): if $LatT^{(k)} \subseteq LatA^{(k)}$ for all $k \in \mathbb{Z}_+$, then $A \in AlgT$. The more general problem of determining all reflexive subalgebras of $[X]$ has so far defied solution.

Notations. $\mathbb{Z}_+ := \{n : n \in \mathbb{Z}, n \geqslant 0\}$; $\mathbb{I}_k$— denotes the identity in $\mathbb{C}^k$, $\mathbb{O}_k := 0 \cdot \mathbb{I}_k$; span$E$ is the closed linear span of the set $E \subseteq X$; supp f is a support of a function $f(x)$; $r * f$ stands for the convolution of functions $r,\ f \in L_1[0,1]\ :\ (r * f)(x) := \int_0^x r(x-t)f(t)dt$; 1_k denotes function $\mathbb{1}$ in the space $W_p^k[0,1]$.

As usual $W_p^k[0,1]$ stands for the Sobolev space : $f \in W_p^k[0,1]$ if f has $k-1$ absolutely continuous derivatives and $f^{(k)} \in L_p[0,1]$. It is a Banach space with respect to the norm

$$||f||_{W_p^k[0,1]} = \left[\sum_{i=0}^{k-1} |f^{(i)}(0)|^p + \int_0^1 |f^{(k)}(t)|^p\,dt\right]^{1/p}.$$

$W^k_{p,0}[0,1] = \{f \in W^k_p[0,1] \ : \ f(0) = \cdots = f^{(k-1)}(0) = 0\}$. We put also $W^0_p[0,1] = W^0_{p,0}[0,1] := L_p[0,1]$ and $E^k_l := \{f \in W^k_p[0,1] \ : f(0) = \ldots = f^{(k-l-1)}(0) = 0\}$ for $l \in \{0,\ldots,k-1\}$ and $E^k_k := W^k_p[0,1]$.

Let J^α_k and $J^\alpha_{k,l}$ denotes the complex powers of the integration operator $(Jf)(x) = \int\limits_0^x f(t)dt$ defined on Sobolev spaces $W^k_p[0,1]$ and E^k_l correspondingly. In what follows we assume that either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $Re\alpha > 0$. Under this assumption the operator J^α_k is well defined on $W^k_p := W^k_p[0,1]$.

The spectral properties of the operator J^α_k. have been investigated in [5]. In particular, descriptions of the lattices $LatJ^\alpha_k$ and $HyplatJ^\alpha_k$ and the set $CycJ^\alpha_k$, as well as the algebras $\{J^\alpha_k\}'$, $\{J^\alpha_k\}''$ and $AlgJ^\alpha_k$ have been obtained in [5].

In the paper under consideration, we prove the reflexivity of the operator $J^\alpha_k \oplus J^\alpha_{k+s}$ and present the description of $Alg(J^\alpha_k \oplus J^\alpha_{k+s})$.

1.Algebra of the operator $J^\alpha_k \oplus J^\alpha_{k+s}$.

To obtain a description of $Alg(J^\alpha_k \oplus J^\alpha_{k+s})$ we need a description of $AlgJ^\alpha_k$ from [5].

Proposition 1. ([5]) The following are true :

1) If either $\alpha = 1$ or $k = 1$, then $AlgJ^\alpha_k = \{J^\alpha_k\}''$ (in particular, $AlgJ_k = \{J_k\}''$);

2) If $1 < \alpha \leqslant k-1$, then $AlgJ^\alpha_k = \{T = cI + R \ : \ c \in \mathbb{C},\ R \in Alg_0J^\alpha_k\}$, where

$$Alg_0J^\alpha_k = \{R \ : \ Rf = r * f,\ r \in W^{k-1}_p[0,1],\ r^{(j)}(0) = 0 \ for\ j \neq i\alpha - 1,\ i \leqslant [\tfrac{k-1}{\alpha}]\}:$$

3) If $k \geqslant 2$ and $Re\alpha \geqslant k - \frac{1}{p}$, then
$AlgJ^\alpha_k = \{T = cI + R \ : \ c \in \mathbb{C},\ Rf = r * f,\ r \in W^{k-1}_{p,0}[0,1]\}$.

Theorem 1. *$Alg(J^\alpha_k \oplus J^\alpha_{k+s}) = \{T \oplus T : T = cI + R, c \in \mathbb{C}, Rf = r * f\}$, where $r \in W^{k+s-1}_p[0,1]$, $r^{(i)}(0) = 0$, for $i \neq j\alpha - 1$, $j \leqslant [\frac{k+s-1}{\alpha}]$.*

Доказательство. Let $T_1 \oplus T_2 \in Alg(J^\alpha_k \oplus J^\alpha_{k+s})$. Then $T_1 \in AlgJ^\alpha_k$, $T_2 \in AlgJ^\alpha_{k+s}$. Hence by Proposition 1

$$(T_1 f)(x) = c\mathbb{I} + r_1 * f, \qquad r_1 \in W^{k-1}_p[0,1],$$

$$(T_2 f)(x) = c\mathbb{I} + r_2 * f, \quad r_2 \in W^{k+s-1}_p[0,1], \quad r_2^{(i)}(0) = 0, \text{ for } i \neq j\alpha - 1, \quad j \leqslant [\frac{k+s-1}{\alpha}].$$

Let

$$E = \{\begin{pmatrix} f \\ f \end{pmatrix} : f \in W^{k+s}_p[0,1]\}. \tag{1}$$

Then $E \in Lat(J^\alpha_k \oplus J^\alpha_{k+s}) \subset Lat(T_1 \oplus T_2)$. Setting $f = 1_{k+s}$ one obtains

$$(T_1 \oplus T_2)\begin{pmatrix} 1_{k+s} \\ 1_{k+s} \end{pmatrix} = \begin{pmatrix} c + r_1(x) * 1_{k+s} \\ c + r_2(x) * 1_{k+s} \end{pmatrix} \in E.$$

Hence $c + r_1(x) * 1_{k+s} \equiv c + r_2(x) * 1_{k+s}$ and $r_1(x) \equiv r_2(x) \in W^{k+s-1}_p[0,1]$. □

2. Reflexivity of the operator $J^\alpha_k \oplus J^\alpha_{k+s}$.

Lemma 1. *Let $A_1 \in [X_1]$, $A_2 \in [X_2]$ and A_1 is isometrically equivalent to the operator A_2, i.e. $UA_1 = A_2U$ with some isometric operator $U : X_1 \to X_2$. Let also $A = A_1 \oplus A_2$ and $B \in AlgLatA$. Then*

1) $B = B_1 \oplus B_2$, where $UB_1 = B_2U$. 2) $B_1 \in \{A_1\}'$, $B_2 \in \{A_2\}'$.

Доказательство. 1) The splitting of B into direct sum of B_1 and B_2 is obvious. In order to prove the equality $UB_1 = B_2U$ we consider subspace $E \subseteq X_1 \oplus X_2$ given by: $E = grU = \{\begin{pmatrix} f \\ Uf \end{pmatrix} : f \in X_1\}$. Since $E \in LatA$, $BE = \{\begin{pmatrix} B_1 f \\ B_2 U f \end{pmatrix} : f \in X_1\} \subseteq E$. The last inclusion yields $\begin{pmatrix} B_1 f \\ B_2 U f \end{pmatrix} = \begin{pmatrix} B_1 f \\ U B_1 f \end{pmatrix}$ for all $f \in X_1$, and hence the first assertion is proved.

2) In order to prove the second assertion we consider subspace $E' = \{\begin{pmatrix} f \\ UA_1 f \end{pmatrix} : f \in X_1\}$. Then

$$AE' = \{\begin{pmatrix} A_1 f \\ A_2 U A_1 f \end{pmatrix} : f \in X_1\} = \{\begin{pmatrix} A_1 f \\ U A_1 A_1 f \end{pmatrix} : f \in X_1\} \subseteq E',$$

$$BE' = \{\begin{pmatrix} B_1 f \\ B_2 U A_1 f \end{pmatrix} : f \in X_1\} = \{\begin{pmatrix} B_1 f \\ U B_1 A_1 f \end{pmatrix} : f \in X_1\}.$$

Since $BE' \subseteq E'$, $UB_1A_1 f = UA_1B_1 f$ for all $f \in X_1$ and hence $B_1 \in \{A_1\}'$.

Further : $B_2A_2 = B_2UA_1U^{-1} = UB_1A_1U^{-1} = UA_1B_1U^{-1} = UA_1U^{-1}UB_1U^{-1} = A_2B_2$. □

For further considerations we recall several facts from [5].

Proposition 2. ([5]) The operator $J^\alpha_{k,l}$ defined on E^k_l is isometrically equivalent to the operator J^α_{k-l} defined on $W^{k-l}_p[0,1]$. In particular the operator $J^\alpha_{k,0}$ defined on $W^k_{p,0}[0,1]$ is isometrically equivalent to the operator J^α_0 defined on $W^0_{p,0}[0,1] := L_p[0,1]$.

Доказательство. It is clear that the operator $U_l = \frac{d^l}{dx^l} : E^k_l \to W^{k-l}_p[0,1]$, isometrically maps E^k_l on $W^{k-l}_p[0,1]$. Moreover,

$$U_l^{-1} = U_l^* = J^l \ : \ W^{k-l}_p[0,1] \to E^k_l.$$

The assertion follows now from the identity $J^\alpha_{k,l} = U_l^{-1} J^\alpha_{k-l} U_l$. □

Proposition 3. ([5]) Let either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $Re\alpha > k - \frac{1}{p}$ and J^α_k be the operator J^α defined on $W^k_p[0,1]$. Then $R \in \{J^\alpha_k\}'$ if and only if

$$(Rf)(x) = \frac{d}{dx} \int_0^x r(x-t) f(t) dt, \quad r \in W^k_p[0,1].$$

In particular, $\{J^\alpha_k\}'$ is commutative algebra and does not depend on α.

Corollary 1. *Let $R \in \{J^\alpha_k\}'$. Then $R \in AlgLatJ^\alpha_k$ if and only if $R \in AlgJ^\alpha_k$.*

Доказательство. Let $R \in AlgLatJ^\alpha_k$ and $m = k - 1 - \alpha[(k-1)/\alpha]$. In order to prove the inclusion $R \in AlgJ^\alpha_k$ one consider subspace

$$E_m = \{f : f \in W^k_p[0,1],\ f^{(m)}(0) = f^{(m+\alpha)}(0) = \cdots == f^{(k-1-\alpha)}(0) = f^{(k-1)}(0) = 0\} \in LatJ^\alpha_k.$$

Then

$$(Rf)^{(k-1)}(0) = r(0) f^{(k-1)}(0) + r'(0) f^{(k-2)}(0) + \cdots + r^{(k-1)}(0) f(0) = 0.$$

Setting $f(x) = x^l$, $l \in \{0, 1, \ldots, k-1\} \setminus \{m, m+\alpha, \ldots, k-1\}$ one obtaines that $r^{(k-1-l)}(0) = 0$ which was has to be proved. □

Theorem 2. *The operator $J^\alpha_k \oplus J^\alpha_{k+s}$ is reflexive for each $s \in \mathbb{Z}_+$.*

Доказательство. To prove the inclusion $AlgLat(J_k^\alpha \oplus J_{k+s}^\alpha) \subseteq Alg(J_k^\alpha \oplus J_{k+s}^\alpha)$ we consider the block matrix representations of the operators $J_k^\alpha \oplus J_{k+s}^\alpha$ and $T \in AlgLat(J_k^\alpha \oplus J_{k+s}^\alpha)$:

$$J_k^\alpha \oplus J_{k+s}^\alpha = \begin{pmatrix} J_k^\alpha & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & J_{k,k+s}^\alpha & * \\ \mathbb{O} & \mathbb{O} & * \end{pmatrix}, \qquad T = A_k \oplus A_{k+s} = \begin{pmatrix} A_k & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & A_{k,k+s} & * \\ \mathbb{O} & \mathbb{O} & * \end{pmatrix}$$

with respect to the direct sum decomposition :

$$W_p^k[0,1] \oplus W_p^{k+s}[0,1] = W_p^k[0,1] \oplus E_k^{k+s} \dotplus span\{1, x, \ldots x^{s-1}\} := W_p^k[0,1] \oplus E_k^{k+s} \dotplus X_s.$$

So

$$A_k \in [W_p^k[0,1]], \quad A_{k+s} \in [W_p^{k+s}[0,1]], \quad A_{k,k+s} \in [E_k^{k+s}], \quad C \in [X_s], \quad B \in [X_s, E_k^{k+s}].$$

By Proposition 2 A_k is isometrically equivalent to the operator $A_{k,k+s} : A_{k,k+s} = U_s^{-1} A_k U_s$, where $U_s = \frac{d^s}{dx^s}$ and hence by Lemma 1 : $A_k \in \{J_k^\alpha\}'$. In view of this Proposition 3 implies

$$(A_k f)(x) = \frac{d}{dx} \int_0^x r(x-t) f(t) dt, \quad r, \ f \in W_p^k[0,1].$$

Further we consider subspace $E \in Lat(J_k^\alpha \oplus J_{k+s}^\alpha)$ of the form (1). Then

$$T \begin{pmatrix} f \\ f \end{pmatrix} = \begin{pmatrix} A_k f \\ A_{k+s} f \end{pmatrix} \in E, \quad f \in W_p^{k+s}[0,1].$$

Hence

$$A_{k+s} f = A_k f = \frac{d}{dx} \int_0^x r(x-t) f(t) dt \in W_p^{k+s}[0,1], \qquad f \in W_p^{k+s}[0,1].$$

It follows that $r(x) = A_k \mathbb{1} \in W_p^{k+s}[0,1]$. Thus, by Theorem 1 : $T \in Alg(J_k \oplus J_{k+s})$. One completes the proof by applying Corollary 1. □

Corollary 2. *Operator J_k^α is 2-reflexive.*

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