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ON DIRECT AND INVERSE SPECTRAL PROBLEM FOR GENERALIZED JACOBI MATRICES

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A new class of generalized Jacobi matrices is introduced. Every generalized Nevanlinna function $m(z)$, which is a solution of an indefinite moment problem is proven to be the m-function of a unique generalized Jacobi matrix. The method we use is based on the step-by-step Schur process of solving the indefinite moment problem.

Ключевые слова: Jacobi matrix, m-function, generalized Nevanlinna function, inverse spectral problem, Schur algorithm

1. Introduction

A real tridiagonal matrix $H = (h_{ij})_{i,j=0}^N$ ($h_{ij} = 0$ if $|i-j| > 1$) is said to be a Jacobi matrix if $h_{i,i+1} = h_{i+1,i} > 0$. It is well known (see [1, 3]) that the spectra of matrices H and $H_{[1,N]} = (h_{ij})_{i,j=1}^N$ interlace. Conversely, given two sets $\{\lambda_j\}_{j=0}^N$, $\{\nu_j\}_{j=1}^N$ of real numbers, such that

$$\lambda_0 < \nu_1 < \lambda_1 < \cdots < \nu_N < \lambda_N$$

there is a unique Jacobi matrix H such that $\sigma(H) = \{\lambda_j\}_{j=0}^N$ and $\sigma(H_{[1,N]}) = \{\nu_j\}_{j=1}^N$. The standard way to solve this inverse problem is to reduce it to some moment problem, use the Gramm-Shmidt orthogonalization procedure in the space of polynomials, and then the needed Jacobi matrix is recovered as a matrix representation for the multiplication operator (see [1]). The other approach is based on the Schur step-by-step process of solving the moment problem associated with the Jacobi matrix (see [10], [8]).

The first method was applied in [11] to a class of generalized Jacobi matrices associated with indefinite moment problems. Since the orthogonalization procedure in an indefinite inner product space leads to the so-called almost orthogonal polynomials [11] which are not uniquely defined one cannot expect the uniqueness result for the corresponding generalized Jacobi matrix. In the present paper we give another definition of generalized Jacobi matrix (by reducing the number of free parameters) and apply the Schur step-by-step process to the corresponding indefinite moment problem in order to solve the inverse problem for generalized Jacobi matrix. This approach leads to more exact results, in particular, the generalized Jacobi matrices is recovered uniquely by the spectral data of its m-function. We essentially use the results of the paper [4], where the Schur step-by-step algorithm of solving indefinite moment problem was elaborated. These results are closely related to the Schur algoritm for generalized Schur functions investigated recently in [2].

2. Generalized Jacobi matrix and its m-function

Let $p(\lambda) = p_n\lambda^n + \cdots + p_1\lambda + p_0$, $p_n = 1$ be a monic scalar polynomial of degree n. Let us associate to the polynomial p its symmetrizator E_p and the companion matrix C_p, given by

$$E_p = \begin{pmatrix} p_1 & \cdots & p_n \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ p_n & & \mathbf{0} \end{pmatrix}, \quad C_p = \begin{pmatrix} 0 & \dots & 0 & -p_0 \\ 1 & & \mathbf{0} & -p_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & -p_{n-1} \end{pmatrix}. \tag{1}$$

Definition 1. (cf. [11]) Let p_j be real monic polynomials of degree n_j

$$p_j(\lambda) = \lambda^{n_j} + p_{n_j-1}^{(j)}\lambda^{n_j-1}\cdots + p_1^{(j)}\lambda + p_0^{(j)},$$

and let $\varepsilon_j = \pm 1$ $(j = 0,\dots,N)$. The tridiagonal block matrix

$$H = \begin{pmatrix} A_0 & \widetilde{B}_0 & & \mathbf{0} \\ B_0 & \ddots & \ddots & \\ & \ddots & \ddots & \widetilde{B}_{N-1} \\ \mathbf{0} & & B_{N-1} & A_N \end{pmatrix} \tag{2}$$

where $A_j = C_{p_j}$ and $n_{j+1}\times n_j$ matrices B_j and $n_j \times n_{j+1}$ matrices $\widetilde{B}_j$ are given by

$$B_j = \begin{pmatrix} 0 & \dots & b_j \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}, \quad \widetilde{B}_j = \varepsilon_j\varepsilon_{j+1}\begin{pmatrix} 0 & \dots & b_j \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix} \quad (b_j > 0,\ j = 0,\dots,N-1). \tag{3}$$

will be called a generalized Jacobi matrix (GJM) associated with the polynomials $\{\varepsilon_j p_j\}_{j=0}^N$.

Let $n+1 = \sum_{j=0}^{N} n_j$. Define a $(n+1)\times(n+1)$ matrix G by the equality

$$G = \operatorname{diag}(G_0,\dots,G_N),\ G_j = \varepsilon_j E_{p_j}^{-1}\ (j = 0,\dots N) \tag{4}$$

and let $\ell_2^{(n+1)}(G)$ be the space of $n+1$ vectors with the inner product

$$\langle x, y\rangle = (Gx, y) \quad (x, y \in \ell_2^{(n+1)}). \tag{5}$$

Let us set a standard basis in $\ell_2^{(n+1)}(G)$ by the equalities

$$e_{j,k} = \{\delta_{l,\sum_{i=0}^{j-1} n_i + k}\}_{l=0}^{n} \quad (j = 0,\dots,N;\ k = 0,\dots,n_j-1),\ e := e_{0,0}.$$

Proposition 1. The GJM H defines a cyclic symmetric operator in $\ell_2^{(n+1)}(G)$.

In the space $\mathbb{C}[n]$ of polynomials of formal degree n there is a basis of polynomials of the first kind $\{P_{j,k}(\lambda)\}_{k=0,\dots,n_j-1}^{j=0,\dots,N}$ such that the multiplication operator in this basis is given by the matrix H. Straightforward calculations show that $P_{j,0}$ turn out to be solutions of the following equations, where $b_{-1} = 0$, $b_N = 1$:

$$\varepsilon_j\varepsilon_{j-1}b_{j-1}P_{j-1,0}(\lambda) - p_j(\lambda)P_{j,0}(\lambda) + b_jP_{j+1,0} = 0 \quad (j = 0,\dots,N). \tag{6}$$

Denote by $H_{[j,m]}$ the shortened GJM corresponding to the basis vectors $\{e_{i,k}\}_{k=0,\dots,n_i-1}^{i=j,\dots,m}$ $(0 \le j \le m \le N)$. The following connection between the polynomials of the first kind and the shortened Jacobi matrices in the classical case can be found in [3].

Proposition 2. Polynomials $P_{j,0}$ can be found by the formulas

$$P_{j+1,0}(\lambda) = (b_0 \dots b_j)^{-1} \det(\lambda - H_{[0,j]}) \quad (j = 0, \dots, N; k = 0, \dots, n_j - 1) \tag{7}$$

$$P_{j,k}(\lambda) = \lambda^k P_{j,0}(\lambda) \quad (j = 0, \dots, N; k = 0, \dots, n_j - 1). \tag{8}$$

Definition 2. The function $m(\lambda) = \langle (H - \lambda)^{-1} e, e \rangle$ $(e := e_{0,0})$ will be called the m-function of the GJM H.

Remind ([11]) that the generalized Nevanlinna class $\mathbf{N}_\kappa$ $(\kappa \in \mathbb{Z}_+)$ consists of functions $\varphi(z)(= \overline{\varphi(\bar{z})})$ meromorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$, such that for every choice of z_i in the domain of holomorphy $\rho(\varphi)$ of the function $\varphi(z)$ the matrix $\left(\frac{\varphi(z_i) - \overline{\varphi(z_j)}}{z_i - \bar{z}_j} \right)_{i,j=1}^n$ has at least κ (and for some choice of z_i exactly κ) negative eigenvalues.

Proposition 3. The m-function of the GJM H belongs to the class $\mathbf{N}_\kappa$, where $\kappa = \operatorname{ind}_-(G)$, and it can be found by the formula

$$m(\lambda) = -\varepsilon_0 \frac{\det(\lambda - H_{[1,N]})}{\det(\lambda - H)}. \tag{9}$$

Define the function $m(\lambda, j)$ by the equality

$$m(\lambda, j) = \langle (H_{[j,N]} - \lambda)^{-1} e_{j,0}, e_{j,0} \rangle \quad (j = 0, \dots, N). \tag{10}$$

Clearly, $m(\lambda, 0) = m(\lambda)$, $m(\lambda, N) = -\dfrac{\varepsilon_N}{p_N(\lambda)}$. In view of Proposition 3 the function $m(\lambda, j)$ belongs to the class N_{κ_j}, where $\kappa_j = \sum\limits_{i=j}^{N} \operatorname{ind}_-(G_i)$ and is given by

$$m(\lambda, j) = -\varepsilon_j \frac{\det(\lambda - H_{[j+1,N]})}{\det(\lambda - H_{[j,N]})} \quad (j = 0, \dots, N - 1). \tag{11}$$

Proposition 4. The functions $m(\lambda, j)$ and $m(\lambda, j+1)$ are connected by the following Ricatti equation (see [8] for the case $\kappa = 0$).

$$m(\lambda, j) = -\varepsilon_j \frac{1}{p_j(\lambda) + \varepsilon_j b_j^2 m(\lambda, j+1)} \quad (j = 0, \dots, N - 1). \tag{12}$$

3. m-FUNCTION AND INDEFINITE MOMENT PROBLEM

Define in the linear space $\mathcal{H} = \mathbb{C}_n[\lambda]$ an indefinite inner product

$$\langle P_{j,k}, P_{j',k'} \rangle_{\mathcal{H}} = (G e_{j,k}, e_{j',k'}) \quad (j, j' = 0, \dots, N; k = 0, \dots, n_j - 1; k' = 0, \dots, n_{j'} - 1). \tag{13}$$

Proposition 5. The m-function of the GJM H has the following asymptotics

$$m(\lambda) \sim -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \dots - \frac{s_{2n}}{\lambda^{2n+1}} \quad (\lambda = iy, y \to +\infty), \tag{14}$$

where $s_{i+j} = \langle \lambda^i, \lambda^j \rangle$, the Hankel matrix $S_n = (s_{i+j})_{i,j=0}^n$ is nondegenerate and $(\kappa :=) \operatorname{ind}_-(S_n) = \operatorname{ind}_-(G)$.

It follows from Proposition 5 that $m(\lambda)$ is a solution of the following indefinite moment problem (see [5], [6], [11])

Problem M(s, n, κ). *Given are a nonnegative integer κ and a sequence $\mathrm{s} = \{s_j\}_{j=0}^{2n}$ of real numbers, such that the matrix $S_n = (s_{i+j})_{i,j=0}^n$ is nondegenerate. Find a function $\varphi \in \mathbf{N}_\kappa$, such that*

$$\varphi(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \dots - \frac{s_{2n}}{\lambda^{2n+1}} + o\left(\frac{1}{\lambda^{2n+1}}\right) \quad (\lambda = iy, y \to +\infty). \tag{15}$$

As is known (see [5], [6]), the problem $\mathbf{M}(\mathbf{s}, n, \kappa)$ is solvable, if

$$\operatorname{ind}_-(S_n) \leq \kappa. \tag{16}$$

Let n_0 be the least natural such that $\det S_{n_0-1} \neq 0$ and let $\kappa_0 = \operatorname{ind}_-(S_{n_0-1})$, $\mathbf{s}^{(0)} = \{s_j\}_{j=0}^{2n_0-2}$. The problem $\mathbf{M}(\mathbf{s}^{(0)}, n_0 - 1, \kappa)$ is elementary in a sense that it defines the first step in the solution of the problem $\mathbf{M}(\mathbf{s}, n, \kappa)$. The step-by-step algorithm for the problem $\mathbf{M}(\mathbf{s}, n, \kappa)$ was elaborated in [4]. Let us set

$$P_{n_0}(\lambda) = \frac{1}{\det S_{n_0-1}} \begin{vmatrix} s_0 & \dots & s_{n_0} \\ \vdots & \ddots & \vdots \\ s_{n_0-1} & \dots & s_{2n_0-1} \\ 1 & \dots & \lambda^{n_0} \end{vmatrix}, \quad \varepsilon_0 = \operatorname{sign} s_{n_0-1}. \tag{17}$$

Theorem 1 ([4]). *Assume that $n_0 - 1 < n$. Every solution φ of the problem $\mathbf{M}(\mathbf{s}, n, \kappa)$ admits the representation*

$$\varphi(\lambda) = -\frac{s_{n_0-1}}{P_{n_0}(\lambda) + \varepsilon_0 \varphi_1(\lambda)}, \tag{18}$$

where $\varphi_1 \in N_{\kappa-\kappa_0}$. Moreover, φ is a solution of the problem $\mathbf{M}(\mathbf{s}, n, \kappa)$, if and only if φ_1 is a solution of the problem $\mathbf{M}(\mathbf{s}^{(1)}, n - n_0, \kappa - \kappa_0)$, where the Hankel matrix $S^{(1)} = (s_{i+j}^{(1)})_{i,j=0}^{n-n_0}$ is given by the equality

$$S_{n-n_0}^{(1)} = |s_{n_0-1}|(T_{n_0} S_n^{-1} T_{n_0}^{\top})^{-1}, \tag{19}$$

and T_{n_0} is $(n - n_0 + 1) \times (n + 1)$ matrix

$$T_{n_0} = \begin{pmatrix} 0 & \dots & 0 & s_{n_0-1} & \dots & s_{n-1} \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & s_{n_0-1} \end{pmatrix} \tag{20}$$

Let now $m(\lambda)$ be the m-function of a GJM H of the order $n+1$. Then $m(\lambda)$ is a solution of the indefinite moment problem $\mathbf{M}(\mathbf{s}, n, \kappa)$, where $s = \{s_i\}_{i=0}^{2n}$ and κ are defined in Proposition 5. Moreover, it turns out that the polynomial P_{n_0} coincides with the polynomial p_0 in the Ricatti equation (12). This observation complements the statement of Proposition 4.

Proposition 6. The m-function $m(\lambda)$ and the function $m(\lambda, 1)$ are connected by the equality

$$m(\lambda) = -\varepsilon_0 \frac{1}{p_0(\lambda) + \varepsilon_0 b_0^2 m(\lambda, 1)}. \tag{21}$$

Moreover, $m(\lambda, 1)$ is a solution of the induced moment problem $\mathbf{M}(\mathbf{s}^{(1)}, n - n_0, \kappa - \kappa_0)$, where the sequence $\mathbf{s}^{(1)} = \{s_i^{(1)}\}_{i=0}^{2(n-n_0)}$ is defined by (19).

4. Inverse problems for generalized Jacobi matrices

When applying Proposition 6 to a rational function $\varphi = \frac{p}{q}$ $(\deg p < \deg q)$ one obtains a decomposition of φ into a continuous fraction

$$\varphi(\lambda) = -\frac{\varepsilon_0}{p_0(\lambda) -} \; \frac{\varepsilon_0 \varepsilon_1 b_0^2}{p_1(\lambda) -} \; \dots \; - \frac{\varepsilon_{N-1} \varepsilon_N b_{N-1}^2}{p_N(\lambda)}. \tag{22}$$

Then the GJM H is recovered by p_j, ε_j and b_j via (2) and (3).

Theorem 2. *Let φ be a proper rational real function of the class $\mathbf{N}_\kappa$. Then there is a unique finite GJM H such that the corresponding m-function $m(\lambda)$ is proportional to $\varphi(\lambda)$.*

Corollary 1. *Let $\{\lambda_i\}_{i=0}^n$ and $\{\nu_i\}_{i=n_0}^n$ be two disjoint sets in $\mathbb{C}$ and let each be symmetric with respect to $\mathbb{R}$, $\varepsilon_0 = \pm 1$, $C(\lambda) = \prod_{i=n_0}^{n} (\lambda - \nu_i)$, $D(\lambda) = \prod_{i=0}^{n} (\lambda - \lambda_i)$ and let $-\varepsilon_0 C(\lambda)/D(\lambda) \in \mathbf{N}_\kappa$. Then there is a unique finite generalized Jacobi matrix H such that $\{\lambda_i\}_{i=0}^n$ are eigenvalues of the matrix H and $\{\nu_i\}_{i=n_0}^n$ are eigenvalues of the matrix $H_{[1,N]}$.*

Let now H be an infinite GJM, and assume that there is an N_0 such that $H_{[N_0+1,\infty]}$ is an infinite Jacobi matrix in the classical sence. Let $\mathcal{H}(\mathbf{s})$ be a completion of the space of polynomials $\mathbb{C}[\lambda]$ with respect to the inner product $\langle \lambda^i, \lambda^j \rangle = s_{i+j}$ $(i, j = 0, 1, 2, \dots)$, where s_i are given by $s_i = \langle H^i e, e \rangle$. Then the operator H can be considered as a matrix representation of the multiplication operator Λ in the basis $P_{j,k}$ in $\mathcal{H}$ $(j = 0, 1, \dots;\ k = 0, \dots, n_j - 1)$. Associated with the sequence $\mathbf{s} = \{s_i\}_{i=0}^\infty$ is the following indefinite moment problem

Problem $\mathbf{M}(\mathbf{s}, \kappa)$. *Given are a nonnegative integer κ and a sequence $\mathbf{s} = \{s_j\}_{j=0}^{2n}$ of real numbers, such that the matrix $S_n = (s_{i+j})_{i,j=0}^n$ is nondegenerate for all n large enough. Find a function $\varphi \in \mathbf{N}_\kappa$, which has the asymptotic expansion* (15) *for all $n \in \mathbb{N}$.*

As was shown in [11] the problem $\mathbf{M}(\mathbf{s}, \kappa)$ is solvable if and only if the Hankel matrices S_n have κ negative eigenvalues for all n large enough. The problem $\mathbf{M}(\mathbf{s}, \kappa)$ is called *determinate* if it has a unique solution $\varphi \in \mathbf{N}_\kappa$. A necessary and sufficient condition for the problem $\mathbf{M}(\mathbf{s}, \kappa)$ to be determinate is the self-adjointness of the operator Λ in $\mathcal{H}$ or, equivalently, the self-adjointness of the operator H in $\ell_2^\infty(G)$. Let us say that the infinite generalized Jacobi matrix H has *type D* if the corresponding operator H is self-adjoint in $\ell_2^\infty(G)$. In this case the operator H is cyclic and e is a generating vector for H. One can define the m-function of a GJM H of type D as in Definition 2.

Proposition 7. Let H be an infinite generalized Jacobi matrix H of type D. Then the m-function of H is a solution of a determinate indefinite moment problem $\mathbf{M}(\mathbf{s}, \kappa)$, where s_j are given by $s_j = \langle H^j e, e \rangle$ $(j = 0, 1, 2, \dots)$ and $\kappa = \operatorname{ind}_- G$.

The properties of $m(\cdot)$ in Proposition 7 completely characterize the GJM H.

Theorem 3. *Let $m(\lambda)$ be an $\mathbf{N}_\kappa$-function such that m is a solution of a determinate indefinite moment problem $\mathbf{M}(\mathbf{s}, \kappa)$, where $\kappa \in \mathbb{Z}_+$ and $\mathbf{s} = \{s_j\}_{j=0}^\infty$ is a sequence of real numbers. Then there is a unique GJM H of type D with the m-function proportional to $m(\lambda)$.*

As a corollary one obtains the following analog of the Stone theorem [1].

Corollary 2. *Every cyclic self-adjoint operator in a Pontryagin space is unitary equivalent to a unique GJM of type D.*

The following corollary extends the result of [7] to the case of GJM.

Corollary 3. *Let $\{\nu_i\}_{i=n_0}^\infty$ and $\{\lambda_j\}_{j=0}^\infty$ $(\lambda_j \neq \nu_i)$ be zeros of two entire functions of minimal exponential type $C(\lambda)$ and $D(\lambda)$, and let $m(\lambda) = -kC(\lambda)/D(\lambda) \in \mathbf{N}_\kappa$ be a solution of a determinate moment problem $\mathbf{M}(\mathbf{s}, \kappa)$ for some $\kappa \in \mathbb{Z}_+$ and $k \in \mathbb{C}$. Then there is a unique GJM H of type D with the discrete spectrum $\{\lambda_i\}_{i=0}^\infty$ and such that $\sigma(H_{[1,\infty]}) = \{\nu_i\}_{i=n_0}^\infty$.*

The statement is immediate from Theorem 3 and the fact that an entire function of minimal exponential type is characterized by its zeros up to a multiplicative constant.

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