Ученые записки Таврического национального университета им. В. И. Вернадского
Серия "Математика. Механика. Информатика и Кибернетика"N 1 (2002) 100 – 104
УДК 517.98

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ON NORMAL OSCILLATIONS OF A VISCOUS STRATIFIED FLUID

The spectrum of normal oscillations, basis properties of eigenfunctions and other questions were studied.

Let a rigid immovable vessel be partially filled with a heavy viscous incompressible stratified fluid (with coefficient of dynamical viscosity $\mu > 0$). We assume that in an equilibrium state the density of a fluid is a function of the vertical variable x_3, i.e., $\rho_0 = \rho_0(x_3)$. In this case the gravitational field with constant acceleration $\vec{g} = -g\vec{e}_3$ acts on the fluid, here $g > 0$ and $\vec{e}_3$ is a unit vector of the vertical axis Ox_3, which is directed opposite to $\vec{g}$. Let Ω be a domain filled with a fluid in equilibrium state, S be a rigid wall of the vessel adherent to the fluid, Γ be a free surface of a fluid. We take the origin O of Cartesian coordinate system $Ox_1x_2x_3$ on Γ. Then the equation of the surface Γ has the form $x_3 = 0$.

Let us consider the basic case of stable stratification of the fluid:

$$0 < N^2_{min} \le N^2(x_3) \le N^2_{max} = N_0^2 < \infty, \qquad N^2(x_3) = -\frac{g\rho_0'(x_3)}{\rho_0(x_3)}, \quad \rho_0(0) > 0, \tag{1}$$

where $N^2(x_3)$ is square frequency of buoyancy.

In equilibrium state the pressure in a fluid is distributed under the law:

$$p_0 = p_0(x_3) = p_0(0) - g\int_0^{x_3} \rho_0(\tau)d\tau. \tag{2}$$

Consider small motions of a fluid near equilibrium state. We denote by $p = p(x,t)$, where $x = (x_1, x_2, x_3) \in \Omega$, the difference of a pressure field from equilibrium field (2), and by $\rho = \rho(x,t)$ the difference of density field from initial field $\rho_0(x_3)$. We denote also the small velocity field in a fluid by $\vec{u}(x,t)$ and a vertical displacement of a free surface from equilibrium one by $\zeta(\hat{x},t)$, $\hat{x} = (x_1,x_2) \in \Gamma$. In the sequel, we suppose the unknown function $\vec{u}(x,t)$, $p(x,t)$, $\rho(x,t)$ and $\zeta(\hat{x},t)$ are infinitesimal of the first order.

Consider linearized Navier-Stokes equations which describe small motion of a viscous stratified fluid, and also boundary conditions and initial data. The initial boundary value problem has the form (see, for instance, [1], [3]):

$$\begin{gathered}
\frac{\partial\vec{u}}{\partial t} = \rho_0^{-1}(x_3)(-\nabla p - g\rho\vec{e}_3 + \mu\Delta\vec{u}) + \vec{f}(x,t) \quad (\text{in}\,\Omega\,), \\
\operatorname{div}\vec{u} = 0, \quad \frac{\partial\rho}{\partial t} + \nabla\rho_0\cdot\vec{u} = 0 \quad (\text{in}\,\Omega\,), \\
\vec{u} = \vec{0} \quad (\text{on}\,S\,), \quad \frac{\partial\zeta}{\partial t} = u_n = \vec{u}\cdot\vec{n} \quad (\text{on}\,\Gamma\,), \quad \int_\Gamma \zeta d\zeta = 0, \\
\mu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) = 0 \quad (\,i=1,2;\,\text{on}\,\Gamma\,), \quad -p + 2\mu\frac{\partial u_3}{\partial x_3} = -g\rho_0(0)\zeta \quad (\,\text{on}\,\Gamma\,), \\
\vec{u}(x,0) = \vec{u}^0(x), \quad \rho(x,0) = \rho^0(x) \quad (\,x\in\Omega\,), \quad \zeta(\hat{x},0) = \zeta_0(\hat{x}) \quad (\,\hat{x}\in\Gamma\,).
\end{gathered} \tag{3}$$

Fist let us introduce the following functional spaces.

a) Denote by $\vec{L}_2(\Omega, \rho_0)$ the space of vector-functions with inner product

$$(\vec{u}, \vec{v}) = \int_\Omega \rho_0(x_3)\vec{u}(x)\overline{\vec{v}(x)}d\Omega.$$

For this space the orthogonal decomposition $\vec{L}_2(\Omega, \rho_0) = \vec{J}_0(\Omega, \rho_0) \oplus \vec{G}_{h,S}(\Omega, \rho_0) \oplus \vec{G}_{0,\Gamma}(\Omega, \rho_0)$ holds ([3]). Here

$$\vec{J}_0(\Omega, \rho_0) = \{\vec{u} \in \vec{L}_2(\Omega, \rho_0) : \operatorname{div}\vec{u} = 0\,(\text{in}\,\Omega), \quad u_n := \vec{u}\cdot\vec{n} = 0\,(\text{on}\,\partial\Omega)\},$$

$$\vec{G}_{h,S}(\Omega, \rho_0) = \{\,\vec{v} \in \vec{L}_2(\Omega, \rho_0) : \vec{v} = \rho_0^{-1}\nabla p, \quad \vec{v}\cdot\vec{n} = 0\,(\text{on}\,S),\ \nabla\cdot\vec{v} = 0\,(\text{in}\,\Omega), \int_\Gamma p d\Gamma = 0\,\},$$

$$\vec{G}_{0,\Gamma}(\Omega, \rho_0) = \{\vec{w} \in \vec{L}_2(\Omega, \rho_0) : \vec{w} = \rho_0^{-1}\nabla\varphi, \quad \varphi = 0(\text{on}\,\Gamma)\}.$$

b) Denote by $L_2(\Gamma)$ the space of function with inner product $(\zeta, \eta)_0 = \int_\Gamma \zeta(\hat{x})\overline{\eta(\hat{x})}d\Gamma$.

Denote by $\Im_2(\Omega)$ the space of scalar-function with inner product

$$(\varphi, \psi) = g^2 \int_\Omega [\rho_0(x_3)N^2(x_3)]^{-1}\varphi\overline{\psi}d\Omega.$$

Further, consider auxiliary boundary value problems and corresponding operators.

Problem 1. Solve the following boundary value problem in unknown function $p_1(x)$:

$$\nabla\cdot(\rho_0^{-1}\cdot\nabla p_1) = 0 \quad (\text{in}\,\Omega\,), \quad \rho_0^{-1}(x_3)\cdot\nabla p_1\cdot\vec{n} = 0 \quad (\text{on}\,S\,),$$

$$\rho_0^{-1}(0)p_1 = \psi \quad (\text{on}\,\Gamma\,), \quad \int_\Gamma \psi d\Gamma = 0.$$

Let $p_1(x)$ be a solution of Problem 1 for $\psi \in H_\Gamma^{\frac{1}{2}}$ ($H_\Gamma^{\frac{1}{2}} = H^{\frac{1}{2}}(\Gamma) \cap H_0$; $H_0 = L_2(\Gamma) \ominus \{1_\Gamma\}$ and $H^{\frac{1}{2}}(\Gamma)$ is the space of Sobolev-Slobodetskij [1]). Then $\rho_0^{-1}(x_3)\nabla p_1 \in \vec{G}_{h,S}(\Omega, \rho_0)$ and $\rho_0^{-1}(x_3)\nabla p_1 = G\psi$, where $G : H_\Gamma^{\frac{1}{2}} \longrightarrow \vec{G}_{h,S}(\Omega, \rho_0)$ is linear bounded operator.

Problem 2. Solve the following boundary value problem in unknown function $\vec{u} \in \vec{J}_{0,S}^1(\Omega, \rho_0)$ and $\rho_0^{-1}\nabla p_2 \in \vec{G}(\Omega, \rho_0)$:

$$\rho_0^{-1}(x_3)\nabla p_2 - P_{0,S}(\rho_0^{-1}\mu\Delta\vec{u}) = \vec{f}_1, \quad \operatorname{div}\vec{u} = 0\,(\text{on}\,\Omega\,),$$

$$\vec{u} = \vec{0}\,(\text{on}\,S\,), \quad \mu(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}) = 0\,(\,i = 1, 2, \text{on}\,\Gamma\,), \quad -p_2 + 2\mu\frac{\partial u_3}{\partial x_3} = 0\,(\text{on}\,\Gamma\,),$$

where $P_{0,S}$ is orthoprojector on the subspace $\vec{J}_{0,S}(\Omega, \rho_0) := \vec{J}_0(\Omega, \rho_0) \oplus \vec{G}_{h,S}(\Omega, \rho_0)$.

The solution of this problem can be written in the form $\vec{u} = \mu^{-1}A^{-1}\vec{f}$, where the operator A is self-adjoint and positive definite (strictly positive) operator, and has the following properties:

1. $D(A) \subset D(A^{\frac{1}{2}}) = \vec{J}_{0,S}^1(\Omega, \rho_0) \subset \vec{J}_{0,S}(\Omega, \rho_0)$, $\overline{D(A)} = \vec{J}_{0,S}(\Omega, \rho_0)$.
2. Operator A has a discrete spectrum $\{\lambda_n(A)\}_{n=1}^\infty$ with accommodation point $+\infty$.
3. Inverse operator A^{-1} is compact and positive and acts in the space $\vec{J}_{0,S}(\Omega, \rho_0)$.

Using the operators defined above and auxiliary boundary value problems, we can formulate the problem (3) as Cauchy problem for differential operator equation in Hilbert space:

$$\frac{d}{dt}\begin{pmatrix}\vec{u}\\ \zeta\\ \rho\end{pmatrix} + \begin{pmatrix}\mu A & gG & C\\ -\gamma_n & 0 & 0\\ -C^* & 0 & 0\end{pmatrix}\begin{pmatrix}\vec{u}\\ \zeta\\ \rho\end{pmatrix} = \begin{pmatrix}\vec{f}_{0,S}\\ 0\\ 0\end{pmatrix} \tag{4}$$

$$(\vec{u}(0), \zeta(0), \rho(0))^t = (\vec{u}^0, \zeta^0, \rho^0)^t, \tag{5}$$

$$(\vec{u}, \zeta, \rho)^t \in \vec{J}_{0,S}(\Omega, \rho_0) \oplus H_0 \oplus \Im_2(\Omega) =: \mathcal{H},$$

where $C^*u := -\nabla\rho_0 \cdot \vec{u}$, $C\rho := P_{0,S}(\rho_0^{-1}(x_3)g\rho\vec{e}_3)$ and $\|C\| = \|C^*\| \leq N_0$, γ_n is trace operator of normal velocity component: $\gamma_n\vec{u} := u_n = \vec{u}\cdot\vec{n}|_\Gamma$ $(\vec{u} \in \vec{J}^1_{0,S}(\Omega, \rho_0))$.

Lemma 1. *a) The operator γ_n can be expanded to operator $\tilde{\gamma}_n$ with the domain $D(\tilde{\gamma}_n) = \{\vec{v} \in \vec{J}_{0,S}(\Omega, \rho_0) : \tilde{\gamma}_n\vec{v} \in L_{2,\Gamma}\}$ and in this case operator $\tilde{\gamma}_n$ is operator adjoint to the operator G from problem1: $\tilde{\gamma}_n = G^*$. b) For operators A and $\tilde{\gamma}_n$ the following inclusions hold:*

$$D(A) \subset D(A^{1/2}) = \vec{J}^1_{0,S}(\Omega, \rho_0) \subset D(\tilde{\gamma}_n).$$

We connect with the problem (4) the operator matrix

$$\mathcal{A}_0 = \begin{pmatrix} A & G & C \\ -G^* & 0 & 0 \\ -C^* & 0 & 0 \end{pmatrix}$$

which has dense in $\mathcal{H}$ domain $D(\mathcal{A}_0) = D(A) \oplus D(G) \oplus \mathfrak{L}_2(\Omega)$. It turns out that the operator $\mathcal{A}_0$ is not closed and therefore is not maximal accretive one.

Put $y(t) := (\vec{u}, \zeta, \rho)^t = e^{at}(\vec{v}, \eta, \sigma)^t$. Then we have the following Cauchy problem from (4)

$$dy/dt + \mathcal{A}_a y = f(t), \quad y(0) = y^0, \tag{6}$$

where $\mathcal{A}_a = \mathcal{A}_0 + a\mathcal{I}$, $\mathcal{I}$ is the identity operator in $\mathcal{H}$.

Theorem 1. *The closure $\mathcal{A} := \overline{\mathcal{A}_a}$ of the operator $\mathcal{A}_a$ is a maximal accretive operator. In addition*

$$D(\mathcal{A}) = \{(\vec{v}, \eta, \sigma)^t \in \mathcal{H} : \vec{v} \in D(A_a^{\frac{1}{2}}), \vec{v} + A_a^{-\frac{1}{2}}Q_1^*\eta \in D(A_a)\}, \tag{7}$$

$$\mathcal{A} = \begin{pmatrix} A_a^{\frac{1}{2}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & Q_1^* & Q_2^* \\ -Q_1 & aI & 0 \\ -Q_2 & 0 & aI \end{pmatrix} \cdot \begin{pmatrix} A_a^{\frac{1}{2}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{8}$$

where $A_a = A + aI$, $Q_1 := G^*A_a^{-\frac{1}{2}}$, $Q_2 := C^*A_a^{-\frac{1}{2}}$.

Consider the homogeneous problem

$$dy/dt + \mathcal{A}y = 0, \tag{9}$$

and its solution of the form

$$y(t) = y \cdot \exp(-\lambda t), \quad y \in \mathcal{H}, \quad \lambda \in \mathbb{C}. \tag{10}$$

We shall call solution (10) by normal oscillations. Substituting function (10) into (9), we obtain the spectral problem

$$\mathcal{A}y = \lambda y, \quad y \in D(\mathcal{A}). \tag{11}$$

Here $\mathcal{A}$ is the matrix operator defined by formulae (7), (8). We shall call $\mathcal{A}$ the operator associated with initial boundary value problem (3).

The operator $\mathcal{A}$ has the following properties.

Lemma 2. *The operator $\mathcal{A}$ has bounded inverse one $\mathcal{A}^{-1}$ such that $\|\mathcal{A}^{-1}\| \leq a^{-1}$. The spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ belongs to the domain*

$$\{\lambda \in \mathbb{C} : Re\lambda \geq a\}. \tag{12}$$

In this the point $\lambda = a$ is infinite multiple eigenvalue and some set of equilibrium states of viscous stratified fluid corresponds to it: $\vec{v} = 0$, $\eta = 0$, $\sigma = \sigma(x_3) \in L_2(x_{3,min}, x_{3,max})$.

The further investigation is based on the theory of linear operators that are self-adjoint in Hilbert space with indefinite metric [2]. Let us introduce the following notations: $\mathfrak{K}_+ := \vec{J}_{0,S}(\Omega, \rho_0)$, $\mathfrak{K}_- := H_0 \oplus \mathfrak{L}_2(\Omega)$.

Lemma 3. *1. The operator matrix $\mathcal{A}$ and its inverse $\mathcal{A}^{-1}$ are $\mathcal{J}$-selfadjoint operators with $\mathcal{J} := diag(I_{\mathfrak{K}_+}, I_{\mathfrak{K}_-})$, where $I_{\mathfrak{K}_+}$ and $I_{\mathfrak{K}_-}$ are identity operators in $\mathfrak{K}_+$ and $\mathfrak{K}_-$, respectively. Spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ is symmetrical with respect to the real axis and is located in domain* (12).
2. Essential (limiting) spectrum $\sigma_{ess}(\mathcal{A})$ of the operator $\mathcal{A}$ coincides with the set $\{a\} \cup \{\infty\}$.
3. Eigenelements corresponding to infinite multiple eigenvalue $\lambda = a$ are negative and has no associate elements.

Lemma 4. *Problem* (11) *has countable set of (finite-multiple) positive eigenvalues $\{\lambda_k^+\}_{k=1}^{\infty}$ with a unique accumulation point $\lambda = +\infty$ and eigenelements $\{y_k^+\}_{k=1}^{\infty}$, $y_k^+ = (\vec{v}_k^+, \eta_k^+, \sigma_k^+)^t \in \mathcal{H}$, such that the set of projections $\{\vec{v}_k^+\}_{k=1}^{\infty}$ onto $\vec{J}_{0,S}(\Omega, \rho_0)$ form Riesz basis with a finite defect in the space $\vec{J}_{0,S}(\Omega, \rho_0)$. This Riesz basis is a p_0-basis (with a finite defect) in $\vec{J}_{0,S}(\Omega, \rho_0)$ for $p_0 > 3/4$.*

Lemma 5. *Problem* (11) *has countable set (finite multiple) positive eigenvalues $\{\lambda_k^-\}_{k=1}^{\infty}$, $\lambda_k^- > a > 0$, with unique accumulation point $\lambda = a$ and eigenelements $\{y_k^-\}_{k=1}^{\infty}$, $y_k^- = (\vec{v}_k^-, \eta_k^-, \sigma_k^-)^t \in \mathcal{H}$, such that the set of projections $(\eta_k^-, \sigma_k^-)^t$ onto $H_0 \oplus \mathfrak{L}_2(\Omega)$ form Riesz basis with a finite defect in the space $H_0 \oplus \mathfrak{L}_2(\Omega)$. This Riesz basis is a p_0-basis (with a finite defect) in $H_0 \oplus \mathfrak{L}_2(\Omega)$ for $p_0 > 3/4$.*

We denote by $F_0(\mathcal{A})$ the closure of linear hull of eigenelements of the operator $\mathcal{A}$ corresponding to the finite multiple eigenvalues, and by $F(\mathcal{A})$ we denote the closure of linear hull of root elements, respectively.

As a corollary of Lemma 4 and Lemma 5, we have the following result.

Theorem 2. *For the operator $\mathcal{A}$ from problem* (11), *the following assertions hold:*

1. $dim(F(\mathcal{A})/F_0(\mathcal{A})) < \infty$.

2. $\mathcal{H} = F(\mathcal{A})$, i.e., the closure of linear hull of root elements of the operator $\mathcal{A}$ coincides with the whole $\mathcal{H} = \vec{J}_{0,S}(\Omega, \rho_0) \oplus H_0 \oplus \mathfrak{L}_2(\Omega)$.

3. $\mathcal{H} = F_0(\mathcal{A})$ iff there are no adjoint elements corresponding to the nonreal eigenvalue λ of the operator $\mathcal{A}$, i.e., $\mathfrak{L}_\lambda(\mathcal{A}) = ker(\mathcal{A} - \lambda\mathcal{I})$.

4. If $\mathcal{H} = F_0(\mathcal{A})$ (respectively $\mathcal{H} = F(\mathcal{A})$), then there exists nearly J-orthonormal Riesz basis formed by eigenelements (respectively root elements) of the operator $\mathcal{A}$.

5. If $F_0(\mathcal{A}) = \mathcal{H}$, then J-orthonormal basis in the space $\mathcal{H}$ formed by eigenelements of the operator $\mathcal{A}$ exists iff $\sigma(\mathcal{A}) \subset \mathbb{R}$.

6. Above-mentioned basises can be chosen as p-basises for $p > 3/2$.

It can be proved that the solutions of spectral problem (11) are directly connected with the solutions of the problem in the form a some operator pencil.

Theorem 3. *Let λ_0 be a finite multiple eigenvalue of the operator $\mathcal{A}$ and elements $y_0, y_1, \ldots, y_k$ be the chain of eigenelement and associated to it, $y_j = (\vec{v}_j, \eta_j, \sigma_j)^t$, $j = \overline{0, k}$. Then $\varphi_0, \varphi_1, \ldots, \varphi_k$, $\varphi_j = A^{\frac{1}{2}}\vec{v}_j$, form the chain of eigenelement and associated to it (in M.V. Keldysh's sense) for the operator pencil:*

$$L(\lambda) = I - \lambda A^{-1} - \lambda^{-1}(B + E), \tag{13}$$

$$B := \tilde{Q}_1^*\tilde{Q}_1 \in \mathfrak{S}_\infty, \quad E := \tilde{Q}_2^*\tilde{Q}_2 \in \mathfrak{S}_\infty, \quad \tilde{Q}_1 := G^*A^{-\frac{1}{2}}, \quad \tilde{Q}_2 := C^*A^{-\frac{1}{2}},$$

and these elements correspond to eigenvalue $\lambda = \lambda_0 - a$.

Inversely, to each chain $\varphi_0, \varphi_1, \dots, \varphi_k$ of eigen- and associated to it elements of the pencil (13) *and eigenvalue $\lambda_0 - a$ there corresponds the chain of root elements $y_1, y_2, \dots, y_k$ of the operator $\mathcal{A}$ and eigenvalue λ_0, where*

$$y_j = (\vec{v}_j, \eta_j, \sigma_j)^t == (A^{-\frac{1}{2}}\varphi_j, \tilde{Q}_1 \sum_{i=0}^{j} (a-\lambda_0)^{i-j-1}\varphi_i, \tilde{Q}_2 \sum_{i=0}^{j} (a-\lambda_0)^{i-j-1}\varphi_i)^t,\ j = \overline{0,k}. \quad (14)$$

As a corollary of Theorem 2 and Theorem 3 we have the following assertion.

Theorem 4. *For the operator $\mathcal{A}$ the following properties are valid.*
1. Nonreal eigenvalues of the operator $\mathcal{A}$ and those real ones to which there correspond associated elements are located in the segment

$$M := \{\lambda \in \mathbb{C} : \ Re\lambda - a \geq \mu(2\|A^{-1}\|)^{-1}, \quad |\lambda - a| \leq 2\mu^{-1}\|gB + E\|\}$$

of complex domain $\mathbb{C}$. To this finite set of eigenvalues there correspond neutral eigenelements of the operator $\mathcal{A}$. Conversely, to neutral eigenelements of the operator $\mathcal{A}$ there correspond eigenvalues that are located in the segment M and for real eigenvalues the operator $\mathcal{A}$ has associated elements.
2. Eigenvalues λ_k^+ corresponding to the positive eigenelements from nonnegative invariant subspace $\mathfrak{L}_+$ are located on interval $(a + 2\mu^{-1}\|gB + E\|, \infty)$. Respectively, eigenvalues λ_k^- corresponding to negative eigenelements from nonpositive invariant subspace $\mathfrak{L}_-$ are located on the interval $[a, a + \mu(2\|A^{-1}\|)^{-1})$.
3. If condition $4\|A^{-1}\| \cdot \|gB + E\| < \mu^2$ holds, then the operator $\mathcal{A}$ has no nonreal eigenvalues, neutral eigenelements and also associated elements. Here the union of normalized eigenelements $\{y_k^+\}_{k=1}^{\infty} \subset \mathfrak{L}_+$ and $\{y_k^-\}_{k=1}^{\infty} \subset \mathfrak{L}_-$ form $\mathcal{J}$-orthonormal basis in the space $\mathcal{H}$.
4. For eigenvalues λ_k^+ and λ_k^- two-side estimates

$$\mu\lambda_k(A) - 2\mu^{-1}\|gB + E\| \leq \lambda_k^+ - a \leq \mu\lambda_k(A), \quad k \in \mathbb{N},$$

$$\mu^{-1}\lambda_k(gB + E) \leq \lambda_k^- - a \leq \mu^{-1}\lambda_k(gB + E)/[1 - 2\mu^{-2}\lambda_k(gB + E)\|A^{-1}\|],$$

and asymptotic formulas

$$\lambda_k^+ = \mu\lambda_k(A) + O(1) = \mu c_a^{-\frac{2}{3}} k^{\frac{2}{3}}[1 + o(1)] \quad c_a = (3\pi^2)^{-1} \int_{\Omega} [\rho_0(x_3)]^{\frac{3}{2}}\, d\Omega,$$

$$\lambda_k^- = a + \mu^{-1}\lambda_k(gB + E)[1 + o(1)] \quad (k \to \infty), \text{ hold.}$$

Author is thankful to prof. Kopachevsky N. D. for statement of the problem and useful discussions.

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Поступила в редакцию 21.03.2002