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SOLONOUKHA O. V.

ON PARTIAL COERCIVE MONOTONE TYPE PROBLEMS

For some noncoercive in classical sense problems it is sufficiently to localize such problem on some convex closed set with a nonempty interiority with the "acute angle"conditions on a boundary. This set may be not a ball. Using last results of set-valued analysis we propose a new "Equilibrium theorem". As applications we consider a localization method for some partial differential problems. As examples the problems with weighted p-Laplacians are analyzed.

Key words: Equilibrium theorem, "acute angle"condition, coercivity, partial coercivity, generalized pseudomonotone operator, partial subdifferential, radially semicontinuous operator of semibounded variation

AMS 35R35, 35R70

1. INTRODUCTION

The studying of the majority of the monotone type problems is based on Leray-Schauder theorem. Thus, we have to localize the problem of such a type on some bounded set. Usually we use the so-called property of coercivity (values of operator tend to infinity sufficiently quickly as the argument tends to ∞). The surjectivity of mappings was proved for many classes of generalized pseudomonotone operators with this property (see works by H.Bresis, F Browder, P.Hess, J.-L.Lions, I.V.Skrypnik, V.S.Mel'nik, V.Barbu, A.C.Kartsatos etc.) We propose condition on some convex closed set where the problem's solution exists (outside of this set the behavoir of operator values is not considered). This condition is the "acute angletype.

In previous works by autor the idea of perturbation was used. But for perturbed method we need the construction of compensating operator and some estimates.

2. EQUILIBRIUM THEOREM AND EXISTENCE OF SOLUTIONS FOR INCLUSIONS

First we consider the equilibrium theorem in finite–dimensional euklid space F. Let $Conv(F)$ be a totality of all nonempty convex closed sets from the space F, 2^F be a totality of all sets, A be a set–valued operator, $\mathrm{Dom}(A)$ be a set where the value of operator is not an empty set, $\overline{\mathrm{co}}Z$ be a convex clozure of Z. Let us define the upper support function for A by the formula $[A(y),\xi]_+ = \sup\limits_{d\in A(y)} \langle d,\xi\rangle$.

Definition 1. The mapping $A : Y \to 2^F$ is called *strong* if $\mathrm{Dom}(A) = F$.

Definition 2. The mapping $A : F \to 2^F$ is *upper semicontinuous* if for any $\varepsilon > 0$ and $y \in \mathrm{Dom}(A)$ there exists $\delta > 0$ such that $A(z) \subset A(y) + B_\varepsilon(0)$ $\forall z \in B_\delta(y)$, $B_\varepsilon(0)$ is a ball of center 0 and radius ε.

Theorem 1. ("Equilibrium theorem") *Let F be a finite-dimensional space, $D \subset F$ be a convex and closed set with nonempty interior, ∂D be a boundary, $y_0 \in \mathrm{int}D$, $A : \overline{D} \to 2^F$ be a strong upper semicontinuous map, and "acute angle's condition"holds:*

$$[A(y), y - y_0]_+ \geq 0 \qquad \forall y \in \partial D \tag{1}$$

Then there exists $x \in \overline{D}$ *such that* $0 \in \overline{co}A(x)$.

Prof of Equilibrium theorem. Without loss of generality we consider $y_0 = 0$ (else we can substitute y for $y - y_0$). Also we suppose that $A : F \to Conv(F)$ (then $\overline{co}A = A$ (see [6])).

Let us consider set-valued map $G : F \to 2^{\partial D}$ which is defined by formulas

$$G(y) = \begin{cases} \partial D, & \text{as } y = 0, \\ hy, \quad \text{where } h = \sup\{h > 0 : hy \in \partial D\}, & \text{as } y \neq 0. \end{cases}$$

G is strong, moreover, $G_{|y \neq 0}$ is single-valued: $0 \in \text{int}D$, thus, $\|hy\|_F \geq \varepsilon \forall hy \in \partial D$ and $0 < \varepsilon^{-1}\|y\|_F \leq h \leq \varepsilon \|y\|_F^{-1} < \infty$ for $y \neq 0$. If $y_n \to 0$, then for bounded set $\{w_n \in G(y_n)\} \subset \partial D$ there exists a convergent subsequence $w_m \to w \in \partial D = G(0)$. If $y_n \to y \neq 0$, then there exists a subsequence $\{y_k\} \cap \{0\} = \emptyset$, $w_k = h_k y_k$, and $0 < h_k < \infty$ is bounded. We can choose convergent subsequences: $y_m \to y \neq 0$, $h_m \to h \neq 0$. But then $w_m = h_m y_m \to w = hy$. Moreover, $\sup_{d \in G}[A(d(y)), y]_+ = 0$ as $y = 0$, $\sup_{d \in G}[A(d(y)), y]_+ \geq h[A(z), z]_+$, as $y \neq 0$ where $hz = y$, $z \in \partial D$. With cone $P = \{0\}$ the mapping A satisfies conditions of Theorem 2 [8], i.e. $\exists x \in \overline{D}$ such that $0 \in A(x)$.■

Now we can consider infinity–dimensional case.

Let X be a reflexive Banach space, X^* be its topological dual, $\langle \cdot, \cdot \rangle$ be the dual pairing on $X \times X^*$, $Conv(X^*)$ be the totality of all nonempty convex closed sets from the space X^*, $A : X \to Conv(X^*)$ be a strong convex-closed-set-valued mapping. By graph(A) we denote the graph of operator A: graph$(A) = \{(y, w) \in Dom(A) \times X^* : w \in A(y)\}$. Now the upper support function and upper norm for A are defined by the formulas

$$[A(y), \xi]_+ = \sup_{d \in A(y)} \langle d, \xi \rangle, \qquad \|A(y)\|_+ = \sup_{d \in A(y)} \|d\|_{X^*}.$$

Taking into account that the support function defines an operator to within a convex closure of values (see Lemma 1 [6]), this theory holds for any set-valued operator.

Definition 3. A mapping $A : X \to Conv(X^*)$ is said to be *generalized pseudomonotone* if for arbitrary $\{(y_n, w_n)\} \subset graph(A)$ such that $y_n \to y$ weakly in X, $w_n \to w$ weakly in X^* and $\overline{\lim_{n \to \infty}} \langle w_n, y_n - y \rangle \leq 0$, we have $w \in A(y)$ and $\langle w_n, y_n \rangle \to \langle w, y \rangle$.

Definition 4. A mapping $A : X \to Conv(X^*)$ *has the property* $(\mathfrak{M})$ if for arbitrary $\{(y_n, w_n)\} \subset graph(A)$ such that $y_n \to y$ weakly in X, $w_n \to w$ weakly in X^* and $\overline{\lim_{n \to \infty}} \langle w_n, y_n - y \rangle \leq 0$, we have $w \in A(y)$.

Definition 5. A mapping $A : X \to Conv(X^*)$ is said to be *monotone* if for any $\{(y_n, w_n)\} \subset graph(A)$ $(n = 1, 2)$ we have that $\langle w_1 - w_2, y_1 - y_2 \rangle \geq 0$. A monotone mapping is *maximal monotone* if its graph is not subset of some other monotone operator's graph.

Definition 6. A mapping $A : X \to Conv(X^*)$ is said to be *bounded* if an image of bounded set is bounded too. A mapping A is said to be *locally bounded (s–weakly locally bounded)* if for any $z \in X$ there exists $\varepsilon > 0$ such that $\sup_{\zeta \in B_\varepsilon(z) \cap \text{Dom}A} \|A(\zeta)\|_+ \leq N$ (if for any $y_n \to y$ weakly in X there exists the subsequence $\{y_{n_k}\}$ such that $\|A(y_{n_k})\|_+ \leq N$).

Using "Equilibrium theorem"1 we can modify Theorem [10] and Theorem 5 [8]. Note that the proposed theorem is different from Theorem [10] and Theorem 5 [8] since we use different "acute angle's condition": earlier this condition was formulated on ball, we consider this condition on some convex closed set with nonempty interior. Theorem 1 shows that this modification is natural. But for convenience of readers we propose the sketch of proof.

Theorem 2. *Let X be a reflexive Banach space, $A : X \to Conv(X^*)$ be a locally bounded on each finite–dimensional F mapping which has the property $(\mathfrak{M})$. Moreover, there exists some convex closed set $D_r \subset X$ with nonempty interior ($y_0 \in \text{int}D_r$) and with boundary ∂D_r such that for $f \in X^*$ the following estimate holds:*

$$[A(y) - f, y - y_0]_+ \geq 0 \qquad \forall y \in \partial D_r. \tag{2}$$

Then the solution set of inclusion

$$A(y) \ni f, \qquad y \in \overline{D_r}$$

is nonempty and compact.

Sketch. We consider $f = 0$, $y_0 = 0$. In general case we can move the coordinate system: $\widetilde{A}(\widetilde{y}) = A(\widetilde{y}) - f$, $\widetilde{y} = y - y_0$.

Let $F(X)$ be a totality of finite-dimensional subspaces $F \subset X$. For arbitrary $F \in F(X)$ we introduce $I_F : F \to X$ ($\|I_F y_F\|_X = \|y_F\|_F \forall y_F \in F$), $I_F^* : X^* \to F^*$ is a dual operator, $D_{rF} = D_r \cap F$, $A_F = A_{|F} : F \to 2^{X^*}$. And we introduce the auxiliaries operators

$$I_F^* A_F(y) = \bigcup_{d \in A_F} \left\{ \sum_{\{h_i\}} \langle d(y), h_i \rangle h_i \right\} \qquad \forall y \in D_{rF} \equiv D_r \cap F,$$

where $\{h_i\}$ is the basis of F. Since A is locally bounded on F and has the property $(\mathfrak{M})$, then $I_F^* A_F$ is upper semicontinuous (Lemma 1[9]). Hence, by "Equilibrium theorem"1 for each $F \in F(X)$ there exists $y_F \in D_{rF}$ such that $0 \in I_F^* A_F(y_F)$. We can construct the system with the finite intersection property $\{\overline{G}^w_{F_0}\}$, where $\overline{G}^w_{F_0}$ is the weak closure of $G_{F_0} = \bigcup\limits_{F \supset F_0} \Big\{ y_F \in D_{rF} : 0 \in I_F^* A_F(y_F) \Big\}$. Since X is reflexive, then $\exists\, y \in \bigcap\limits_{F \in F(X)} \{\overline{G}^w_F\}$. We obtain $0 \in A_F(y_F)$, $y_F \to y$ weakly in X, and $\overline{\lim}\langle 0, y_F - y \rangle = 0$. Thus, by property $(\mathfrak{M})$ $0 \in A(y)$. □

3. Existence of solutions for variational inequalities

In this section we propose a new sufficient conditions when a variational inequality has at last one solution. With respect to result of [9, 11, 10] we loosen a coercivity and intensify a boundness.

Denote by $N_K(y)$, $N_K^1(y$ the normal cone of the set $K \subset X$ at the point $y \in X$ and the frustum of this cone:

$$N_K(y) := \big\{ g \in X^* : \langle \xi, g \rangle_X \leq 0 \quad \forall \xi \in \bigcup_{h>0} \frac{1}{h}(K - y) \big\},$$
$$N_K^1(y) := \big\{ g \in N_K(y) : \|g\|_{X^*} \leq 1 \big\}.$$

As it is well known, $N_K : X \to 2^{X^*}$ is maximal monotone operator for any closed convex set K (see [3]), i.e. it is monotone and closed in weak-strong topology of $X^* \times X$ or $X \times X^*$ (see [1]).

Lemma 1. *The mapping $\lambda N_K^1 : K \to Conv(X^*)$ is bounded and generalized pseudomonotone for any $\lambda > 0$.*

Доказательство. It is sufficiently to prove for $\lambda = 1$. This map is bounded by construction.

Let $K \ni y_n \to y$ weakly in X, $N_K^1(y_n) \ni w_n \to w$ weakly in X^*, and $\overline{\lim\limits_{n\to\infty}} \langle w_n, y_n - y \rangle \leq 0$. Since $w_n \in N_K(y_n)$ and N_K is maximal monotone, then $w \in N_K(y)$ and $\langle w_n, y_n \rangle \to \langle w, y \rangle$. But $\|w_n\|_{X^*} \leq 1$, i.e. $\|w\|_{X^*} \leq 1$ too. Consequently, $w \in N_K^1(y)$. □

Lemma 2. *Let $A : X \to Conv(X^*)$ has bounded values. Then for any fixed convex closed set $K \subset X$ the element $y \in K$ is the solution of variational inequality*

$$[A(y), v - y]_+ \geq \langle f, v - y\rangle \qquad \forall v \in K \tag{3}$$

iff there exist $(y, \lambda) \in K \times (0, \infty)$ such that

$$A(y) + \lambda N_K^1(y) \ni f. \tag{4}$$

Доказательство. Let $y \in K$ be a solution of variational inequality. Then there exists $w \in A(y)$ such that $\langle w, v - y\rangle \geq \langle f, v - y\rangle \quad \forall v \in K$ (Lemma 4 [9]). Let us assume $d = f - w$. Then, $d \in N_K(y)$ and $\|d\|_{X^*} \leq \|f\|_{X^*} + \|w\|_{X^*} \leq \|f\|_{X^*} + \|A(y)\|_+ \leq \lambda$, i.e. $d \in \lambda N_K^1(y)$. We get the inclusion (4): $f = d + w \in Ly + A(y) + \lambda N_K^1(y)$.

Let $y \in K$ be a solution of inclusion (4). Multiplying by $v - y$, we get

$$\langle f, v - y\rangle = [A(y) + \lambda N_K^1(y), v - y]_+ \leq [A(y), v - y]_+ \, \forall v \in K.$$

□

Thus, we can consider variational inequalities as well as inclusions.

Theorem 3. *Let X be reflexive Banach space; $A : K \to Conv(X^*)$ be a bounded, generalized pseudomonotone operator; $K \subset X$ be a closed convex set. Moreover, one of following conditions holds: A satisfies "acute angle's condition"(1) on some convex closed set D_r where $y_0 \in \operatorname{int}(K \cap D_r)$ or K is bounded. Then the set of solutions for variational inequality (3) is nonempty and weakly compact in $D_r \cap K$.*

Доказательство. It suffices to show that the inclusion (4) is solvable.

By the definition $0 \in N_K(\xi)$ for any $\xi \in K$, $0 \in N_K^1(\xi)$ for any $\xi \in K$, and $[N_K^1(y), y - y_0]_+ = \|y - y_0\|_X > 0$ as $y \in \partial K$. Since

$$[A(y) - f + \lambda N_K^1(y), y - y_0]_+ = [A(y) - f, y - y_0]_+ + \\ +\lambda[N_K^1(y), y - y_0]_+ \geq (-\|A(y)\|_+ - \|f\|_{X^*} + \lambda)\|y - y_0\|_X,$$

then for arbitrary bounded set K it is sufficiently to take $\lambda = \sup\limits_{y \in K} \|A(y)\|_+ + \|f\|_{X^*}$. And we obtain the estimate on ∂K. If K is not bounded, we use "acute angle's condition": there exists $D_r > 0$ such that $[A(y) - f, y - y_0]_+ \geq 0$ as $\|y\|_X \geq R$. Then we take $\lambda = \sup\limits_{y \in K \cap D_r} \|A(y)\|_+ + \|f\|_{X^*}$. Thus, $[A(y) + \lambda N_K^1(y) - f, y - y_0]_+ \geq 0$ as $y \in \partial(K \cap D_r)$. Moreover, the mapping $A + \lambda N_K^1$ is bounded and has the property $(\mathfrak{M})$ as a generalized pseudomonotone mapping (see Proposition 2[11]). By Theorem 2 there exists $\hat{y}$ such that $f \in A(\hat{y}) + \lambda N_K^1(\hat{y})$.

Let $y_n \to y$ weakly in X, y_n satisfy (4). Since $\overline{\lim\limits_{n \to \infty}} \langle f, y_n - y\rangle \leq 0$, using the property $(\mathfrak{M})$ we have that $f \in A(y) + \lambda N_K^1(y)$. Thus, the set of solutions is weakly compact. □

Definition 7. A mapping $A : X \to Conv(X^*)$ is said to be *"+coercive on K* if there exist $y_0 \in K$ and $c : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$[A(y), y - y_0]_+ \geq c(\|y\|_X)\|y - y_0\|_X, \quad c(\gamma) \to \infty \text{ as } \gamma \to \infty.$$

Definition 8. A mapping A is said to be *partially "+coercive on K* if there exist a subspace $Y \subset X$ ($\dim Y \geq 1$), some element $y_0 \in \operatorname{int} K$ and functions $c, \tilde{c} : \mathbf{R}_+ \to \mathbf{R}_+$ such that $c(\gamma) \to \infty$ as $\gamma \to \infty$, $\tilde{c}$ is proper, i.e. $\tilde{c} \neq +\infty$, and

$$[A(y), y - y_0]_+ \geq c(\|\mathrm{pr}_Y y\|_X)\|\mathrm{pr}_Y(y - y_0)\|_X - \tilde{c}(\|\mathrm{pr}_{X \setminus Y} y\|_X))\|\mathrm{pr}_{X \setminus Y}(y - y_0)\|_X.$$

Lemma 3. *If $A : K \to Conv(X^*)$ is partially "+coercive on K, then for any $f \in X^*$ there exists set D_r such that*

$$[A(y) - f, y - y_0]_+ \geq 0 \qquad \forall y \in \partial D_r.$$

Доказательство. We choose $D_r = \{y \in B_R(y_0) : \|\mathrm{pr}_Y y\|_X = r_1, \|\mathrm{pr}_{X\setminus Y} y\|_X) = r_2\}$ where

$$r_1(c(r_1) - \|f\|_{X^*}) \geq r_2(\tilde{c}(r_2) + \|f\|_{X^*}).$$

Since $c(\gamma) \to \infty$ as $\gamma \to \infty$ for sufficient large R this bounded convex set exists. Then on this set the "acute angle's condition"(1) hold. □

Corollary 1. *Let X be reflexive Banach space; $A : K \to Conv(X^*)$ be a bounded, generalized pseudomonotone operator; $K \subset X$ be a closed convex set. Moreover, one of following conditions holds: A is partially "+coercive on K. Then there exist D_r such that set of solutions for variational inequality (3) is nonempty on $D_r \cap K$ and weakly compact.*

4. Application to free boundary partial differential problems

Let $\Omega \subset \mathbf{R}^n$ be a locally Lipschitz open bounded set with the regular boundary $\partial\Omega$, ν be an external normal by $\partial\Omega$, $Dy = (\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n})$. We consider the boundary problem

$$-\sum_{1\leq i\leq n} \frac{\partial}{\partial x_i} a_i(x, y, Dy) = f \qquad \text{a.e. on}\, \Omega, \tag{5}$$

$$\frac{\partial y}{\partial \nu_A} = \sum_{1\leq i\leq n} a_i(x, y, Dy)\cos(x_i, \nu)$$

$$y \geq 0, \quad \frac{\partial y}{\partial \nu_A} \geq 0, \quad y\frac{\partial y}{\partial \nu_A} = 0 \qquad \text{a.e. on}\, \partial\Omega, \tag{6}$$

where $f \in L_q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \in (n, \infty)$. We use also r and r' such that $\frac{1}{r} + \frac{1}{p} = 1$, $\frac{1}{r'} + \frac{3}{p} = 1$.

Let a_i satisfy the following conditions

I) (Caratheodori cond.): $a_i(\cdot, y, \xi)$ are measurable for a.a. $y \in \mathbf{R}^1$, $\xi \in \mathbf{R}^n$ and $a_i(x, \cdot, \cdot)$ are continuos at a.a. $x \in \Omega$; moreover, at a.a. $x \in \Omega$ and for all $y \in \mathbf{R}$, $\xi \in \mathbf{R}^n$ there exists nondecreasing and continuous function $\alpha : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$|a_i(x, y, \xi)| \leq \alpha(|y|)(\sum_{j=1}^n |\xi_j|^{p-1} + h(x)),\ h \in L_q(\Omega).$$

II) for $x \in \Omega$, $y \in \mathbf{R}$, $\xi, \eta \in \mathbf{R}^n$ such that $\xi \neq \eta$

$$\sum_{i=1}^n \big(a_i(t, x, y, \xi) - a_i(t, x, y, \eta)\big)(\xi_i - \eta_i) > 0.$$

III) partial coercivity condition:

$$\sum_{i=1}^n a_i(x, \xi_0, \xi)\xi_i \geq c_0 \sum_{i\in I_1} |\xi_i|^p - \tilde{c}_0 \sum_{i\in I_2} |\xi_i|^p - g(x),$$

where $c_0, \tilde{c}_0 > 0$, $I_1 \cup I_2 = \{0, 1, 2, \cdots, n\}$, $I_1 \neq \emptyset$; $g \in L_1(\Omega)$.

Using condition I) and boundary conditions we obtain the variational inequality

$$\sum_{i=1}^n \int_\Omega a_i(x, y, Dy)\frac{\partial(\xi - y)}{\partial x_i}\, dx \geq \int_\Omega f(\xi - y)\, dx$$

$$\forall \xi \in W_p^{+,1}(\Omega) = \{y \in W_p^1(\Omega) : y_{|\partial\Omega} \geq 0\}.$$

By Lemmas 2.3 and 2.6 [4] corresponding operator is bounded, continuous and generalized pseudomonotone. By property III) we have the partial coercivity of this problem and we can localize oun problem. Thus, problem (5)–(6) has at least one generalized solution $y \in W_p^{+,1}(\Omega)$.

5. Weak solvability of nondivergent problems

Let as early $\Omega \subset \mathbf{R}^n$ be a locally Lipschitz open bounded set with the regular boundary $\partial\Omega$. We consider the boundary problem

$$-\sum_{1\le i\le n} a_i(x,y)\,\frac{\partial}{\partial x_i} b_i(x,y,Dy) = f \qquad \text{a.e. on}\,\Omega, \tag{7}$$

$$\frac{\partial y}{\partial \nu_A} = \sum_{1\le i\le n} a_i(x,y)\, b_i(x,y,Dy)\, \cos(x_i,\nu)$$

$$y \ge 0, \quad \frac{\partial y}{\partial \nu_A} \ge 0, \quad y\frac{\partial y}{\partial \nu_A} = 0 \qquad \text{a.e. on}\,\partial\Omega, \tag{8}$$

where $f \in L_q(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$, $p \in (n,\infty)$. And let $\frac{1}{r_1+r_2}+\frac{1}{p}=1$, $\frac{1}{r_1'+r_2'}+\frac{3}{p}=1$.

Let a_i, b_i satisfy the following conditions

a1) (Caratheodori cond.): $a_i(\cdot,y), b_i(\cdot,y,\xi)$ are measurable for a.a. $y \in \mathbf{R}^1$, $\xi \in \mathbf{R}^n$ and $a_i(x,\cdot)$, $b_i(x,\cdot,\cdot)$ are continuos at a.a. $x \in \Omega$;

b1) at a.a. $x \in \Omega$ and for all $y \in \mathbf{R}$, $\xi \in \mathbf{R}^n$

$$|a_i(x,y)| \le \alpha_i^{1a}(x) + \alpha_i^{2a}(x)|y| \quad \text{or} \quad |a_i(x,y)| \le \alpha_i^{1a}(x) + C^a|y|^{p-1-r_2}$$

$$|b_i(x,\xi_0,\xi)| \le \alpha_i^{1b}(x) + \sum_{0\le j\le n} \alpha_{ij}^{2b}(x)|\xi_j|$$

$$\text{or} \quad |b_i(x,\xi_0,\xi)| \le \alpha_i^{1b}(x) + \sum_{0\le j\le n} C_{ij}^b|\xi_j|^{p-1-r_1}$$

where $\alpha_i^{1a} \in L_{r_1}(\Omega)$, $\alpha_i^{2a} \in L_{r_1'}(\Omega)$, $\alpha_i^{1b} \in L_{r_2}(\Omega)$, $\alpha_{ij}^{2b} \in L_{r_2'}(\Omega)$, $C^a > 0$, $C_{ij}^b > 0$;

c1) if at point $x \in \Omega$ the partial derivatives $\frac{\partial}{\partial x_i}a_i(x,y)$ exist then

$$\left|\frac{\partial}{\partial x_i}a_i(x,y)\right| \le \beta_i^1(x) + \beta_i^2(x)|y| \ \text{ or } \ \left|\frac{\partial}{\partial x_i}a_i(x,y)\right| \le \beta_i^1(x) + C'|y|^{p-2-r_2}$$

where $\beta_i^1 \in L_{r_1}(\Omega)$, $\beta_i^2 \in L_{r_1'}(\Omega)$, $C' > 0$, and $\left|y\frac{\partial}{\partial x_i}a_i(x,y)\right| \le a_i(x,y)\left|\frac{\partial y}{\partial x_i}\right|$;

d1) $a_i(\cdot,y)$ and $a_i(x,\cdot)$ are locally Lipschitz, in particular, $\forall y$ there exists $K(\cdot,y) \in L_{r_1}(\Omega)$ such that

$$|a_i(x,y) - a_i(x,z)| \le K(x,y)|y-z| \qquad \forall z \in B_\varepsilon(x,y)(y), \tag{9}$$

where $B_\varepsilon(\cdot,\cdot)(y)$ is an ε-neighburhood of y;

e1) $a_i(x,y) \ge \tilde{a}_i > 0$ at a.a. $x \in \Omega$ and for all $y \in \mathbf{R}^1$, and $\exists\lambda_{0i} > 0$ such that $a_i(x,(1+\lambda)y) \ge (1+\lambda)a_i(x,y)$ as $\lambda \in [0;\lambda_{0i}]$;

f1) $\sum\limits_{1\le i\le n} b_i(x,y,\xi_1,\cdots,\xi_n)\xi_i \ge c_0\sum\limits_{i\in I_1}|\xi_i|^p - \tilde{c}_0\sum\limits_{i\in I_2}|\xi_i|^p - \varkappa$, where $c_0, \tilde{c}_0, \varkappa > 0$, $I_1 \cup I_2 = \{0,1,2,\cdots,n\}$, $I_1 \ne \emptyset$.

Denote

$$a_{ij}^0(x,y;\delta x_j) = \limsup_{z\to x,\lambda\to+0} \frac{a_i(z+\lambda\delta x_j, y) - a_i(z,y)}{\lambda},$$

$$a_{iy}^0(x,y;h) = \limsup_{\zeta\to y,\lambda\to+0} \frac{a_i(x,\zeta+\lambda h) - a_i(x,\zeta)}{\lambda},$$

then we can construct the integral form (in [12] this construction is more detail)

$$[A(y),\xi-y]_+ = \sum_{1\le i\le n}\int_\Omega a_i(x,y)b_i(x,y,Dy)\frac{\partial(\xi-y)}{\partial x_i}dx +$$

$$+\sum_{1\le i\le n}\int_\Omega a^0_{iy}(x,y;\xi-y)\frac{\partial y}{\partial x_i}\,b_i(x,y,Dy)\,dx+$$
$$+\sum_{1\le i\le n}\int_\Omega b_i(x,y,Dy)\,(\xi-y)a^0_{ix}(x,y;dx)\ge \tag{10}$$
$$\ge\int_\Omega f(\xi-y)dx\qquad \forall\xi\in W^{+,1}_p(\Omega),$$

where $W^{+,1}_p(\Omega)=\{z\in W^1_p(\Omega):z_{|\partial\Omega}\ge 0\}$.

Definition. $y\in W^{+,1}_p(\Omega)$ is called **the weak solution of (7)–(8)** if y satisfies the variational inequality (10).

Analogeously to Theorem 2[12] we prove that $A:W^1_p(\Omega)\to Conv(W^{-1}_q(\Omega))$ is bounded and generalized pseudomonotone. Moreover, using condition f1) we obtain that A is partialy " coercive. (7)–(8) has a weak solution.

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