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STRONG HAMBURGER MOMENT PROBLEM

1. The *strong moment problem* was introduced in [3] and [4]. In this problem, moments $\{s_k\}_{-\infty}^{+\infty}$ of all positive and negative orders are given. A full description of all solutions of the strong Stieltjes moment problem was obtained in [5].

Let us state the strong Hamburger moment problem (SHMP). Given a sequence of real numbers $\{s_k\}_{-\infty}^{+\infty}$. Find a Lebesgue–Stieltjes measure $d\sigma$ on $\mathbb{R}$ such that

$$s_k = \int_{-\infty}^{+\infty} t^k \, d\sigma(t) \quad (k \in \mathbb{Z}). \tag{1}$$

The following questions are considered: solvability, uniqueness and description of all solutions of the problem (1).

2. The answer to the first question is given by

Theorem 1. *The problem* (1) *is solvable if and only if*

$$\sum_{i,j=0}^{n} s_{2m+i+j}\xi_i\xi_j \geq 0 \quad (m \in \mathbb{Z},\ n \in \mathbb{N},\ \xi_k \in \mathbb{R}). \tag{2}$$

3. It is possible that one of the forms

$$\sum_{i,j=0}^{n} s_{2m+i+j}\xi_i\xi_j \quad (m \in \mathbb{Z},\ n \in \mathbb{N}) \tag{3}$$

is not strictly positive. Then the problem is called degenerated. It turns out that in the degenerated case all the forms (3) which order does not exceed some n_0 are strictly positive and all the forms of higher order are degenerated. In this case Theorem 1 can be strengthened.

Theorem 2. *Let SHMP be degenerated and n_0 is defined as above. Then the SHMP has a unique solution $d\sigma$ and the support of $d\sigma$ consists of n_0 points.*

4. Henceforth we assume that all forms (3) are strictly positive. Let us set

$$e_k(t) = t^k \quad (k \in \mathbb{Z},\ t \in \mathbb{R}). \tag{4}$$

Linear combinations of (4) are called quasipolynomials. Let H_0 be the set of all quasipolynomials. Let H be a completion of H_0 with respect to the inner product

$$(e_i, e_j) = s_{i+j} \quad (i, j \in \mathbb{Z}).$$

Let us define a linear operator A_0 in H_0 by the rule

$$A_0 e_k = e_{k+1} \quad (k \in \mathbb{Z}).$$

The closure of A_0 to H is denoted by A. Clearly, H is a Hilbert space and A is a closed symmetric operator in H.

The following theorem describes solutions of the SHMP.

Theorem 3. *Solutions of the problem* (1) *and spectral functions F_t of the operator A (see* [1]*) are in a one-to-one correspondence by the formula*

$$\sigma(t) = (F_t e_0, e_0) \quad (t \in \mathbb{R}).$$

5. Let the numbers $D_n^{(m)}$ be given by

$$D_n^{(m)} = \begin{vmatrix} s_m & s_{m+1} & \dots & s_{m+n} \\ s_{m+1} & s_{m+2} & \dots & s_{m+n+1} \\ \dots & \dots & \dots & \dots \\ s_{m+n} & s_{m+n+1} & \dots & s_{m+2n} \end{vmatrix} \quad (m \in \mathbb{Z},\ n \in \mathbb{N}).$$

The strict positivity of the forms (3) is equivalent to the inequalities

$$D_n^{(2k)} > 0 \quad (k \in \mathbb{Z},\ n \in \mathbb{N}).$$

Let us define the quasipolynomials $\{P_k\}_0^\infty$ of the first kind

$$P_{2k} = \left(D_{2k-1}^{(-2k)} D_{2k}^{(-2k)}\right)^{-\frac{1}{2}} \begin{vmatrix} s_{-2k} & s_{-2k+1} & \dots & s_0 \\ s_{-2k+1} & s_{-2k+2} & \dots & s_1 \\ \dots & \dots & \dots & \dots \\ s_{-1} & s_0 & \dots & s_{2k-1} \\ e_{-k} & e_{-k+1} & \dots & e_k \end{vmatrix} \quad (k \in \mathbb{N}),$$

$$P_{2k+1} = \left(D_{2k}^{(-2k)} D_{2k+1}^{(-2k-2)}\right)^{-\frac{1}{2}} \begin{vmatrix} s_{-2k-1} & s_{-2k} & \dots & s_0 \\ s_{-2k} & s_{-2k+1} & \dots & s_1 \\ \dots & \dots & \dots & \dots \\ s_{-1} & s_0 & \dots & s_{2k} \\ e_{-k-1} & e_{-k} & \dots & e_k \end{vmatrix} \quad (k \in \mathbb{N}). \tag{5}$$

Proposition 8. Quasipolynomials (5) form an orthonormal basis in H.

Let the quasipolynomials of the second kind $\{Q_k\}_0^\infty$ be given by

$$Q_k(t) = \left((A-t)^{-1}(P_k - P_k(t)), e_0\right) \quad (k \in \mathbb{N}).$$

6. Decomposing the functions AP_k by the basis $\{P_k\}_0^\infty$ (cf. [2]), we obtain the following

Proposition 9. The functions $\{AP_k\}_0^\infty$ can be reduced to the form

$$\begin{aligned} AP_0 &= a_0 P_0 + b_0 P_1 + c_0 P_2, \\ AP_1 &= b_0 P_0 + a_1 P_1 + b_1 P_2, \\ AP_{2k} &= c_{2k-2} P_{2k-2} + b_{2k-1} P_{2k-1} + a_{2k} P_{2k} + b_{2k} P_{2k+1} + c_{2k} P_{2k+2} \quad (k = 1, 2, 3, \dots), \\ AP_{2k+1} &= b_{2k} P_{2k} + a_{2k+1} P_{2k+1} + b_{2k+1} P_{2k+2} \quad (k = 1, 2, 3, \dots), \end{aligned}$$

where $\{a_k\}_0^\infty$, $\{b_k\}_0^\infty$, $\{c_k\}_0^\infty$ are sequences of real numbers such that

$$a_{2k+1} = \frac{b_{2k} b_{2k+1}}{c_{2k}},\ c_{2k} > 0,\ c_{2k+1} = 0 \quad (k \in \mathbb{N}).$$

7.

Proposition 10. The operator A is selfadjoint if and only if the following series is divergent

$$\sum_{k=0}^{\infty} |P_k(z)|^2 \quad (\operatorname{Im} z \neq 0). \tag{6}$$

If the series (6) is convergent then A is a symmetric operator with deficiency indices (1, 1) and its defect subspace $\mathfrak{N}_z$ is the span of the vector

$$\sum_{k=0}^{\infty} P_k(z)P_k.$$

8. Let the indeterminate case takes place, that is deficiency indices of A are $(1,1)$. Let us use the formula for the resolvent matrix $W(z)$ of the operator A from [6]. If $a \in \mathbb{R} \setminus \{0\}$ then $W(z)$ takes the form

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix} =$$
$$\begin{pmatrix} -(z-a)\sum_{k=0}^{\infty} Q_k(a)Q_k(z) & 1+(z-a)\sum_{k=0}^{\infty} P_k(a)Q_k(z) \\ -1+(z-a)\sum_{k=0}^{\infty} Q_k(a)P_k(z) & -(z-a)\sum_{k=0}^{\infty} P_k(a)P_k(z) \end{pmatrix} \quad (z \in \mathbb{C} \setminus \mathbb{R}). \quad (7)$$

Theorem 4. *In the intedeterminate case the full description of all solutions of the strong Hamburger moment problem is given by the formula*

$$\int_{-\infty}^{+\infty} \frac{1}{t-z}\, d\sigma(t) = s_0 \cdot \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \quad (\tau \in \widetilde{\mathcal{R}}).$$

Here $\mathcal{R}$ is a class of holomorphic functions $\tau(z)$ on $\mathbb{C}_+$ such that $\operatorname{Im}\tau(z) \geq 0$ for all $z \in \mathbb{C}_+$, and $\widetilde{\mathcal{R}} = \mathcal{R} \cup \{\infty\}$.

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