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RECONSTRUCTION OF A SINGULAR POTENTIAL IN TWO DIMENSIONAL SCHRÖDINGER OPERATOR. BORN APPROXIMATION

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Abstract *We prove that in dimension two potential scattering the leading order singularities (in some special cases - all singularities) of unknown potential are obtained exactly by the Born approximation. The proof is based on the new estimates for the continuous spectrum of the Laplacian in the weighted L^p-spaces and the new estimates for the Green-Faddeev's function in L^p. Using these estimates, we prove the well-known Saito's formula, uniqueness theorem of the reconstruction unknown potential by the scattering amplitude and recovering singularities in the general case, in the backscattering, in the fixed angle scattering and at a fixed energy. We prove also the new asymptotical formula for the Fourier transform of the unknown potential. These estimates allow us to consider the potentials with stronger singularities than in previous publications.*

1. Introduction

Let $H = -\Delta + q(x)$ be a Schrödinger operator in R^2 with the real-valude potential $q(x)$. We assume in this article that the potential belongs to the weighted space $L^2_\sigma(R^2)$ defined by the norm

$$\|q\|_{2,\sigma} = \left(\int_{R^2} (1+|x|)^{2\sigma} |q(x)|^2\, dx\right)^{1/2}, \tag{1}$$

where $\sigma > 1$.

Below we also use the following notations. The space H^t denotes the usual L^2– based Sobolev space and the space W^t_p denotes the L^p– based Sobolev space in R^2.

Under the above assumptions on the potential the Hamiltonian H is a self-adjoint operator in $L^2(R^2)$. The spectrum of this operator consists of a continuous spectrum, filling out the positive real axes (without non-negative eigenvalues), and a possible negative discrete spectrum of the finite multiplicity with zero as the only possible accumulation point. If in addition we suppose the potential has some power decay at the infinity $|q(x)| \le C|x|^{-\mu}$, $|x| > R$, for some $\mu > 2$, then the negative discrete spectrum can be only finite and zero is the point of the continuous spectrum (see [3],[20]). That's why for arbitrary $k \in R, k \neq 0$, we define the scattering solutions of the homogeneous Schrödinger equation

$$(H - k^2)u(x, k) = 0$$

to be the unique solutions of the Lippmann-Schwinger equation

$$u(x, k, \vartheta) = e^{ik(x,\vartheta)} - \int_{R^2} G_k^+(|x-y|)q(y)u(y,k,\vartheta)\, dy,$$

where $\vartheta \in S^1$ and the outgoing fundamental solution of the Helmholtz equation G_k^+ is defined as

$$G_k^+(|x|) = \frac{i}{4} H_0^{(1)}(|k||x|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind and of order 0. The function G_k^+ is the kernel of the integral operator $(-\Delta - k^2 - i0)^{-1}$.

The solutions $u(x,k,\vartheta)$ for $k>0$ admit asymptotically as $|x|\to+\infty$ the representation

$$u(x,k,\vartheta)=e^{ik(x,\vartheta)}+Ce^{ik|x|}k^{-1/2}|x|^{-1/2}A(k,\vartheta',\vartheta)+o\left(\frac{1}{|x|^{1/2}}\right),$$

where $\vartheta'=\frac{x}{|x|}\in S^1$, C is a constant and the scattering amplitude $A(k,\vartheta',\vartheta)$ is defined by

$$A(k,\vartheta',\vartheta)=\int\limits_{R^2}e^{-ik(\vartheta',y)}q(y)u(y,k,\vartheta)\,dy. \tag{2}$$

It what follows we extend A to negative k by $A(k,\vartheta',\vartheta)=\overline{A(-k,\vartheta',\vartheta)}$ to obtain a well-defined scattering amplitude for all $k\in R, k\neq 0$. We use also the fact that $A(k,\vartheta',\vartheta)=A(k,-\vartheta,-\vartheta')$.

We will consider the problem of recovering the singularities of the potential and the potential itself assuming that we know the scattering amplitude $A(k,\vartheta',\vartheta)$ for certain data.

As a different data for the reconstruction of unknown potential $q(x)$ we consider the kernel $G_q(x,y,k)$ of the integral operator $(H-k^2-i0)^{-1}$ which is the solution of the following integral equation:

$$G_q(x,y,k)=G_k^+(|x-y|)-\int\limits_{R^2}G_k^+(|x-z|)q(z)G_q(z,y,k)\,dz. \tag{3}$$

Definition 1. We say that the Hamiltonian H has a resonance at zero if the homogeneous Lippmann-Schwinger equation for $k=0$

$$u(x,0,\vartheta)=-\int\limits_{R^2}G_0^+(|x-y|)q(y)u(y,0,\vartheta)\,dy,$$

where $G_0^+(|x|)=\frac{1}{2\pi}\log\frac{1}{|x|}$, has a non-trivial solution from the space of the continuous functions uniformly vanishing at the infinity.

It follows from (1.2) that for every fixed point $\xi\in R^2$ (see, for example, [20])

$$(Fq)(\xi)=\lim_{k\to+\infty}A(k,\vartheta',\vartheta),\quad \xi=k(\vartheta-\vartheta')$$

and also we have

$$(Fq)(2\xi)=A(k,-\vartheta,\vartheta)+o_k(1),\quad k=|\xi|,\quad \vartheta=\frac{\xi}{|\xi|},$$

where F is the ordinary Fourier transform in R^2. The latter formulas justify the following definitions.

Definition 2. The inverse Born approximation $q_B(x)$ of the potential $q(x)$ is defined as follows:

$$q_B(x):=\frac{1}{32\pi^3}\int\limits_{R\times S^1\times S^1}e^{-ik(\vartheta-\vartheta',x)}A(k,\vartheta',\vartheta)|k||\vartheta-\vartheta'|\,dk\,d\vartheta\,d\vartheta'. \tag{4}$$

Definition 3. The inverse Born backscattering approximation $q_B^b(x)$ of the potential $q(x)$ is defined as follows:

$$q_B^b(x):=\frac{1}{4\pi^2}\int\limits_{R^2}e^{-i(\xi,x)}A\left(\frac{|\xi|}{2},-\frac{\xi}{|\xi|},\frac{\xi}{|\xi|}\right)d\xi. \tag{5}$$

Definition 4. The inverse Born fixed angle scattering approximation $q_B^{\vartheta_0}$ of the potential $q(x)$ is defined as follows:

$$q_B^{\vartheta_0}(x):=\frac{1}{16\pi^2}\int\limits_{R\times S^1}e^{-ik(\vartheta-\vartheta_0,x)}A(k,\vartheta_0,\vartheta)|k||\vartheta-\vartheta_0|\,dk\,d\vartheta, \tag{6}$$

where $\vartheta_0\in S^1$ is fixed.

It is very easy to see that within the Born approximation, the scattering amplitude is simply the Fourier transform of the unknown potential. The weaker the potential, the better is this approximation. But even when the potential is not weak the Fourier transform of a scattering amplitude contains essential information of the potential as was shown in [15] and [17] in two dimensions and in [16] and [18] in three and higher dimensions. In a series of papers starting from 1969 Prosser [19] has shown that the inverse backscattering problem has a unique solution assuming the potential has a small enough weighted Hölder norm. Generic uniqueness results for the backscattering problem in two and three dimensions were obtained by Eskin and Relston [4]-[6]: they proved that the nonlinear operator taking the potential to the backscattering data is an analytic local homomorphism on an open and dense set of an appropriate functional space (some subsets of Sobolev's spaces). Without any smallness assumptions Stefanov [26] has shown that if two compactly supported $L^\infty-$ potentials q_1 and q_2 have the same backscattering data and in addition he has assumed $q_1 \geq q_2$, then in fact $q_1 = q_2$. Stefanov [26]-[27] gave also a simple proof in three dimensions for the generic uniqueness of the backscattering problem and the fixed angle scattering problem in the case of compactly supported potentials belonging to the Sobolev's psace W^4_∞. We recall that in the case of less singularity of the potentials (compare with our case in the present paper) Sun and Uhlmann [30] -[31] have considered related problems in two dimensions with the fixed energy data, while Greenleaf and Uhlmann [8] considered related problems in R^n with the backscattering data. We have to mention here the articles of Novikov [13] and Novikov and Henkin [14] where considered some similar problems for singular potentials and also incoming article of Ruiz [21] about the reconstruction of singularities in the fixed angle scattering problem in two dimensional case for the potentials with compact support from Sobolev's spaces $H^s(R^2)$ for some non-negative s.

In this paper we follow to the ideas of the articles [15]-[18]. The main role in these considerations has played the estimates of the resolvent for the Schrödinger operator at the continuous spectrum (see [24]):

$$\|(H-k^2-i0)^{-1}f\|_{L^4_{-\sigma/2}} \leq \frac{C}{|k|^{1/2}}\|f\|_{L^{4/3}_{\sigma/2}}, \quad q(x)\in L^2_\sigma(R^2), \tag{7}$$

where $\sigma > 1$.

In fact we can obtain more sharp estimate of the resolvent for the Schrödinger operator for such potentials (this estimate follows by interpolation from well-known Agmon's estimates for the Laplacian in the L^2 - weighted spaces [1] and the estimates for the Laplacian which are contained in the Theorem 2.3 of [9] for $n=2$):

$$\|(H-k^2-i0)^{-1}f\|_{L^4_{-\sigma/2}} \leq \frac{C}{|k|^{3/4}}\|f\|_{L^{4/3}_{\sigma/2}}$$

with the same σ and $q(x)$ as in (1.7).

But for our aims we need the estimate for the integral operator $\mathbf{K}$

$$\mathbf{K} := |q|^{1/2}(-\Delta-k^2-i0)^{-1}|q|^{-1/2}q, \quad q(x)\in L^2_\sigma(R^2),$$

in L^2 which it follows from the latter estimate

$$\|\mathbf{K}\|_{L^2\to L^2} \leq \frac{C}{|k|^{3/4}}. \tag{8}$$

Concerning the inverse problem of reconstruction of the singularities of unknown potential by the knowledge of the scattering amplitude at a fixed energy ($k_0^2 \geq 0$) we would like to say that the crucial role in this problem plays the following Green-Faddeev's function:

$$G_z(x) := \frac{1}{4\pi^2}\int_{R^2} \frac{e^{i(x,z+\xi)}}{\xi^2+2(z,\xi)}\,d\xi, \tag{9}$$

where $z\in \mathbf{C}^2$ is two dimensional complex vector and $(z,z)=k_0^2$ and it's mapping in some L^p- spaces (see [25]):

$$\|e^{-i(x,z)}G_z * f\|_{L^\infty(R^2)} \leq \frac{C}{|z|^\gamma}\|f\|_{L^2\cap L^1(R^2)}, \tag{10}$$

where $\gamma < 2/3$.

Due to last estimate we can prove that there exists the special solution to the Schrödinger equation:

$$(-\Delta+q(x)-k_0^2)u(x)=0$$

in the form (this is the "non-physical"solution or the Faddeev's solution [7]):

$$u(x,z)=e^{i(x,z)}(1+R(x,z)), \tag{11}$$

where function $R(x,z)$ satisfies the estimate

$$\|R\|_{L^\infty(R^2)} \le \frac{C}{|z|^\gamma}$$

with γ as in (1.10).

The most important fact here is the knowledge of the scattering amplitude at the fixed energy uniquely determines as a function of $\xi \in R^2$ the following function (see [12])

$$T_q(\xi) := \int_{R^2} e^{i(x,\xi)}(1 + R(x,z))\, dx, \quad |\xi| > 1, \quad T_q(\xi) := 0, \quad |\xi| < 1, \tag{12}$$

where $z = \frac{1}{2}(i\xi + J\xi)$ and $J = \|a_{jl}\|$ is the matrix with $a_{11} = a_{22} = 0, a_{12} = -a_{21} = 1$ and for the simplicity we took $k_0 = 0$. In that case the Born approximation can be has the following form.

Definition 5. The inverse Born fixed energy approximation $q_B^f(x)$ of the potential $q(x)$ is defined as follows:

$$q_B^f(x) := F^{-1}(T_q(\xi)), \tag{13}$$

where F^{-1} is the inverse Fourier transform and the function T_q from (1.12).

The fixed energy inverse problem is well understood in dimensions higher than two (see [7], [10], [28] and [29]). The main result is that the scattering amplitude with a fixed energy uniquely determines a compactly supported bounded potential. However, the problem is not solved in the dimension two (see [11] and [32]). In the articles [30]-[31] Sun and Uhlmann proved that the knowledge of the scattering amplitude at the fixed energy determines the location of the singularity as well as the jumps across the curve of the discontinuity for a compactly supported bounded potential and the reconstruction of singularities for the potentials from $L^p_{comp}(R^2)$ for $p > 2$. We improve these results for the potentials with stronger singularities and without assumption about the compact support of the potential.

The main idea of our considerations consists in the asymptotic expansion for the Born's potential (for all inverse problems which are presented here) in the form: $q_B = \sum_{j=0}^{\infty} q_j$ analogously to the symbol expansions in the pseudodifferential calculus. By noting that q_0 is our unknown potential $q(x)$ the problem is reduced to estimating the smoothness of the higher order terms in the Born's expansions. This expansions is merely the iterations of the Lippmann-Schwinger equation. That's why is so important the estimate (1.8) for to prove the convergence of this series.

We are now in the position to formulate our main results about recovering of the singularities for the Schrödinger operator with singular potentials.

2. Main results

Theorem 1. *([17]) Assume that the potential $q(x)$ belongs to $L^2_\sigma(R^2)$ with $\sigma > 1$ and Hamiltonian H has no resonance at zero. Then*

$$q_B(x) - q(x) \in W^1_\delta(R^2) + H^\beta(R^2), \tag{14}$$

where $\delta < 2$ and $\beta < 1/2$ (if we use "better"estimate for the the resolvent then we can choose $\beta < 1$).

The latter relation allow us assert that the difference between $q_B(x)$ and $q(x)$ belongs to the "smoother"space than unknown potential. This fact means that the leading order singularities of unknown potential are obtained exactly by the Born approximation (1.4). If we suppose that our potential $q(x)$ belongs to the "smoother"space $L^p_{loc}(R^2)$ for $p > 3$ (and even for $p > 2$), then we can prove that the difference $q_B(x) - q(x)$ is a continuous function. It means that in this case we can obtained all singularities of unknown potential by Born approximation. In particular we can reconstruct the boundary of any unknown domain.

Theorem 2. *([15]) Assume that the potential $q(x)$ has bounded support and belongs to the space $H^s(R^2)$ for some $0 < s \le 1$. Then*

$$q_B^b(x) - q(x) - q_1(x) \in H^t(R^2), \tag{15}$$

where $t < \frac{s+1}{2}$ and the first nonlinear term $q_1(x)$ is a continuous function for $1/2 < s \le 1$ and belongs to the space $H^{2s}(R^2)$ for $0 < s \le 1/2$.

From this theorem we obtain the immediate corollary.

Corollary 1. *If a piecewise smooth compactly supported potential $q(x)$ contains the jumps over a smooth curve, then the curve and height function of the jumps are uniquely determined by the backscattering data. Especially, for the potential being the characteristic function of a smooth bounded domain this domain is uniquely determined by the scattering data.*

A potential satisfying the assumptions of the last Corollary is in $H^s_{comp}(R^2)$ for every $s < 1/2$. Thus by Theorem 2, $q^b_B(x) - q(x)$ is in $H^t(R^2)$ for every $t < 3/4$. That's why this Corollary is satisfied.

Theorem 3. *If $q(x)$ as in the Theorem 1 then*

$$q_B^{\vartheta_0}(x) - q(x) \in H^t(R^2). \tag{16}$$

where $t < 1/8$.

Theorem 4. *If $q(x)$ as in the Theorem 1 then*

$$q_B^f(x) - q(x) - q_1(x) \in H^t(R^2). \tag{17}$$

where $t < 1/3$ and first nonlinear term $q_(x)$ (see [31]) is a continuous function.

Theorem 5. (Saito's formula) *([20],[22],[23]) Under the same assumptions for $q(x)$ as in Theorem 1*

$$\lim_{k\to+\infty} k \int\limits_{S^1\times S^1} e^{-ik(\vartheta-\vartheta',x)} A(k,\vartheta',\vartheta)\, d\vartheta\, d\vartheta' = 4\pi \int\limits_{R^2} \frac{q(y)}{|x-y|}\, dy.$$

This limit is valid in the sense of the theory of distributions.

The next theorem is the uniqueness theorem of the reconstruction of unknown potential by the scattering amplitude and it is a simple corollary from Saito's formula.

Theorem 6. *Assume that the potentials $q_1(x)$ and $q_2(x)$ satisfy the conditions of Theorem 1 and the corresponding scattering amplitudes coincide for some sequence $k_j \to +\infty$ and for all $\vartheta, \vartheta' \in S^1$. Then $q_1(x) = q_2(x)$ (in the sense of the theory of distributions).*

It follows from the Saito's formula very interesting connection between the potential $q(x)$ and the scattering amplitude $A(k,\vartheta',\vartheta)$ (see [20])

$$q(x) = \lim_{k\to+\infty} \frac{k^2}{8\pi^2} \int\limits_{S^1\times S^1} e^{-ik(\vartheta-\vartheta',x)} A(k,\vartheta',\vartheta) |\vartheta-\vartheta'|\, d\vartheta\, d\vartheta'.$$

This formula should be understood in the sense of theory of distributions.

It what follows from the proof of Saito's formula in the case when we have the scattering amplitude only with one fixed direction ϑ_0 that

$$\lim_{k\to+\infty} k^{1/2} \int\limits_{S^1} e^{-ik(\vartheta-\vartheta_0,x)} A(k,\vartheta_0,\vartheta)\, d\vartheta = 0.$$

This limit is valid in the sense of uniformly convergence with respect to x.

The new asymptotical formula for the unknown potential which contains the only Green's function of the Hamiltonian is presented in following theorem.

Theorem 7. *([20]) Assume that the potential $q(x)$ satisfies the conditions (1.1) and has special behaviour at the infinity: $|q(x)| \le C|x|^{-\mu}$ for $|x| > R$ with some positive C, R and $\mu > 2$. Then the Fourier transform $F(q)$ of the potential $q(x)$ belongs to the space $L^\infty(R^2) \cap C(R^2)$ and in every point ξ can be calculated by the formula*

$$F(q)(\xi) = \lim_{|x|,|y|,k\to+\infty} 8\pi i k(|x||y|)^{1/2} e^{-ik(|x|+|y|)} (G_q(|x-y|) - G_k^+(x,y,k)).$$

where $\xi = k(\frac{x}{|x|} + \frac{y}{|y|})$ and the function G_q satisfies the integral equation (1.3).

The main results (Theorem 1 and Theorem 2) of this work were obtained together with Lassi Päivärinta from the University of Oulu, Finland. Theorem 1 was also obtained together with Erkki Somersalo from Helsinki University of Technology, Finland. Theorem 2 was obtained together with Petri Ola from the University of Oulu and Theorems 5-7 were obtained together with my Ph.D students, Aleksey G. Razborov from Moscow State University and Melis K. Sagyndykov from the University of Osh, Kyrgyziya.

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