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INVARIANT AND HYPERINVARIANT SUBSPACES OF THE OPERATOR J^α IN THE LIOUVILLE SPACES

1. INTRODUCTION

It is well known ([B], [GLR], [N]) that the Volterra integration operator defined on $L_p[0,1]$ by $J\colon f \to \int_0^x f(t)\,dt$ is unicellular for $p \in [1,\infty)$ and it's lattice of invariant subspaces is anti-isomorphic to the segment $[0,1]$.

Tha same is also true for the complex powers of the integration operator J:

$$J^\alpha\colon f \to \int\limits_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt, \quad \operatorname{Re}\alpha > 0 \tag{1}$$

More precisely, the lattice of invariant and hyperinvariant subspaces of the operator J^α are of the form:

$$\operatorname{Lat} J^\alpha = \operatorname{Hyplat} J^\alpha = \{E_a := \chi_{[a,1]} L_p[0,1]\colon 0 \le a \le 1\} \tag{2}$$

E. Tsekanovskii [Ts] has obtained a description of the lattice $\operatorname{Lat} J_k$ of invariant subspaces of the integration operator $J_k := J$ defined on the Sobolev space $W_2^k[0,1]$.

I. Domanov and M. Malamud [DM1]- [DM2] have described the lattices $\operatorname{Lat} J_k^\alpha$ and $\operatorname{Hyplat} J_k^\alpha$ of invariant and hyperinvariant subspaces of the operator J_k^α defined on $W_p^k[0,1]$ and investigated the operator algebras $\operatorname{Alg} J_k^\alpha$, commutant $\{J_k^\alpha\}'$, and double commutant $\{J_k^\alpha\}''$.

In particular, it is shown in [DM1]-[DM2] that the operator J^α is unicellular on $W_2^k[0,1]$ (with $k \ge 2$) if and only if $\alpha = 1$.

It is also shown in [DM1]-[DM2] that $\operatorname{Hyplat} J_k^\alpha = \operatorname{Hyplat} J_k = \operatorname{Hyplat}^c J_k \cup \operatorname{Hyplat}^d J_k$, where

$$\operatorname{Hyplat}^c J_k = \{E_a\colon 0 \le a \le 1\}, \qquad E_a := \{f\colon f \in W_p^k[0,1], f = 0 \text{ for } x \in [0,a]\} \tag{3}$$

is a continuous chain and $\operatorname{Hyplat}^d J_k = \{E_l^k\}_{l=0}^k$ with $E_k^k := W_p^k[0,1]$ and

$$E_l^k = \{f \in W_p^k[0,1]\colon f(0) = \dots = f^{(k-l-1)}(0) = 0\}, \qquad l \in \{0,1,\dots,k-1\} \tag{4}$$

is a discrete chain.

The following description of the commutant $\{J_k^\alpha\}'\colon R \in \{J_k^\alpha\}' \iff R = I + R_1$, where $R_1 \in \bigcap_{q>1} \sigma_q$ have been also obtained in [DM2].

In the paper under consideration we generalize several results from [DM1]-[DM2] to the case of Liouville space $L_p^s[0,1]$ ($s > 0$). Namely, we describe the lattices $\operatorname{Lat} J_s^\alpha$ and $\operatorname{Hyplat} J_s^\alpha$ of invariant and hyperinvariant subspaces of the operator J_s^α defined on Liouville spaces $L_p^s[0,1]$ and investigate the operator algebras $\operatorname{Alg} J_s^\alpha$, commutant $\{J_s^\alpha\}'$, and double commutant $\{J_s^\alpha\}''$. For integer $s = k \in \mathbb{Z}_+$ our results coinside with that from [DM2]. We follow the method proposed in [DM2].

2. Notations

$W_p^k[0,1]$ stands for the Sobolev space: $f \in W_p^k[0,1]$ if f has $k-1$ absolutely continuous derivatives and $f^{(k)} \in L_p[0,1]$.

Let $s > 0$. Then $k-1 < s \le k$, $k \in \mathbb{Z}_+$. $L_p^s[0,1]$ stands for the Liouville space: $f \in L_p^s[0,1]$ if f the fractional derivative of the order $s-k+1$ belongs to $W_p^k[0,1]$. The space $L_p^s[0,1]$ may be characterized by the following way: $f \in L_p^s[0,1]$ iff it admits a representation

$$f(x) = \sum_{m=1}^{k} c_m \frac{x^{s-m}}{\Gamma(s-m+1)} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} g(t)\, dt$$

where $c_m = f^{(s-m)}(0)$ and $g(x) = f^{(s)}(x)$.

It is a Banach space with respect to the norm:

$$\|f\|_{w_p^s[0,1]} = \left(\sum_{m=1}^{k} |f^{(s-m)}(0)|^p + \int_0^x |f^{(s)}(x)|^p\, dx \right)^{1/p} .$$

We denote by $L_{p,0}^s[0,1] = \{f \in L_p^s[0,1] \colon f^{(s-m)}(0) = 0, 1 \le m \le k\}$.

3. Invariant subspaces and cyclic subspaces of the operator J^α in $W_p^s[0,1]$ and $W_{p,0}^s[0,1]$.

Let $J_{s,0}^\alpha$ and J_s^α stand for the operator J^α defined on $L_{p,0}^s[0,1]$ and $L_p^s[0,1]$ respectively. In what follows we assume that either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\operatorname{Re}\alpha > k - \frac{1}{p}$. Under this assumption the operator J_s^α is well defined on $L_p^s[0,1]$.

Lemma 1. *The operator $J_{s,0}^\alpha$ defined on $L_{p,0}^s[0,1]$ is isometrically equivalent to the operator J_0^α on $L_p[0,1]$*

Definition 1. Let X be a Banach space. An operator $T \in [X]$ is called unicellular if its lattice of invariant subspaces $\operatorname{Lat} T$ is linearly ordered. We will say that $\operatorname{Hyplat} T$ is unicellular if it is linear ordered too.

Proposition 1. Let $\operatorname{Re}\alpha > 0$ and $J_{s,0}^\alpha$ be defined on $L_p^s[0,1]$. Then

$$\operatorname{Lat} J_{s,0}^\alpha = \{E_a^s \colon 0 \le a \le 1\}, \qquad E_a^s = \{f \in L_{p,0}^s[0,1] \colon f(x) = 0 \text{ for } x \in [0,a]\}. \tag{5}$$

and thus $J_{s,0}^\alpha$ is unicellular.

To present a description of $\operatorname{Lat} J_s^\alpha$ we recall a description of $\operatorname{Lat} Q$ for a nilpotent operator $Q \in [\mathbb{C}^k]$.

Theorem 1 ([BF], [GLR]). *If Q is nilpotent of a finite-dimensional vector space V, then*

$$\operatorname{Lat} Q = \bigcup_M \{[M, Q^{-1}M] \colon M \in \operatorname{Lat}(Q|QV)\},$$

where $[M, Q^{-1}M]$ is an interval in the lattice of all subspaces of V. Each interval satisfies the equation

$$\dim Q^{-1} - \dim M = \dim \ker Q.$$

For each bounded operator T on Banach space X, ($T \in [X]$) and $E \in \operatorname{Lat} T$ we denote by $\hat{T}_E$ the quotient operator acting on the quotient space X/E according to the natural rule $\hat{T}\hat{f} = \widehat{(Tf)}$, where $\hat{f}$ stands for a coset $\hat{f} = f + E$.

Theorem 2. *Let π be a quotient map,*

$$\pi\colon L_p^s[0,1] \to X_k := L_p^s[0,1]/L_{p,0}^s[0,1]$$

and $\hat{J}^\alpha$ be the quotient operator on X_k. Then $\operatorname{Lat} J_s^\alpha = \operatorname{Lat}^c J_s^\alpha \cup \operatorname{Lat}^d J_s^\alpha$, *where*

a)

$$\operatorname{Lat}^c J_s^\alpha = \{E_a^s\colon 0 \le a \le 1\}, \qquad E_a^s = \{f \in L_{p,0}^s[0,1]\colon f(x) = 0 \text{ for } x \in [0,a]\}$$

is a "continuous part" of $\operatorname{Lat} J_s^\alpha$*;*

b)

$$\operatorname{Lat}^d J_s^\alpha = \pi^{-1}(\operatorname{Lat} \hat{J}_s^\alpha) = \bigcup_M \pi^{-1}\{[M, (\hat{J}_s^\alpha)^{-1}M]\colon M \in \operatorname{Lat}(\hat{J}_s^\alpha | \hat{J}_s^\alpha M)\}$$

is a "discrete part" of $\operatorname{Lat} J_s^\alpha$.

Here $[M, (\hat{J}_s^\alpha)^{-1}M]$ is a closed interval in the lattice ov all subspaces of X_k. Each interval satisfies the equation

$$\dim(\hat{J}_s^\alpha)^{-1}M - \dim M = d,$$

where $d = \min\{-[-\alpha], k\}$.

One obtains the proof using Lemma 1 and Theorem 1.

Corollary 1. *Let* $0 < s \le 1$. *The operator J_s^α is unicellular in $L_p^s[0,1]$ if* $\operatorname{Re}\alpha > 1 - \frac{1}{p}$. *Moreover*

$$\operatorname{Lat} J_s^\alpha = \operatorname{Lat} J_{s,0}^\alpha \cup L_p^s[0,1] = \{E_a : 0 \le a \le 1\} \cup L_p^s[0,1].$$

Corollary 2. *Let $s > 1$ $(k-1 < s \le k)$ and either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or* $\operatorname{Re}\alpha > k - \frac{1}{p}$. *Then:*

1) the operator J_s^α is unicellular in $L_p^s[0,1]$ if and only if $\alpha = 1$;

2) $\operatorname{Lat} J_s = \operatorname{Lat}^c J_s \cup \operatorname{Lat}^d J_s$, *where* $\operatorname{Lat}^c J_s$ *is defined by* (5) *and*

$$\operatorname{Lat}^d J_s = \{L_{p,0}^s = E_0^s \subset E_1^s \subset \cdots \subset E_k^s := L_p^s[0,1]\}$$
$$E_l^s := \operatorname{span}\{L_p^s, x^{s-1}, \dots, x^{s-l}\}, \quad 1 \le l \le k.$$

Definition 2. Recall that a subspace E of a Banach space X is called a cyclic subspace for an operator $T \in [X]$ if $\operatorname{span}\{T^n E\colon n \ge 0\} = X$. A vector $f \in X$ is called cyclic if $\operatorname{span}\{T^n f\colon n \ge 0\} = X$. The set of all cyclic subspaces of an operator T is denoted by $\operatorname{Cyc}(T)$.

Definition 3. We set

$$\mu_T := \inf_E\{\dim E\colon E \in \operatorname{Cyc}(T)\}.$$

μ_T is called the spectral multiplicity of an operator T in X. Note that μ_T can be ∞.

It is clear that the operator T is cyclic iff $\mu_T = 1$.

The following result immediately follows from Proposition 1.

Proposition 2. Let $\operatorname{Re}\alpha > 0$. Then the operator $J_{s,0}^\alpha$ defined on $L_{p,0}^s[0,1]$ is cyclic. Moreover the following equivalence holds:

$$f \in \operatorname{Cyc} J_{s,0}^\alpha \iff \int_0^\varepsilon |f(x)|^p\,dx > 0 \text{ for all } \varepsilon > 0.$$

Proposition 3. Let $k - 1 < s \le k$.

1) The spectral multiplicity $\mu_{J_s^\alpha}$ of J_s^α is

$$\mu := \mu_{J_s^\alpha} = \begin{cases} \min\{\alpha, k\}, & \alpha \in \mathbb{Z}_+ \setminus \{0\} \\ k, & \alpha \notin \mathbb{Z}_+ \setminus \{0\}. \end{cases} \tag{6}$$

2) The system $\{f_j\}_1^N$ of vectors $f_j \in L_p^s[0,1]$ generates a cyclic subspace for J_s^α if and only if:

i) $N \geq \mu$
ii) $\operatorname{rank} W_\mu\{f_1, \ldots, f_N\}(0) = \mu$, where

$$W_\mu\{f_1, \ldots, f_N\}(x) = \begin{pmatrix} f_1(x) & \ldots & f_N(x) \\ f_1'(x) & \ldots & f_N'(x) \\ \ldots & \ldots & \ldots \\ f_1^{(\mu-1)}(x) & \ldots & f_N^{(\mu-1)}(x) \end{pmatrix}$$

Corollary 3. *Let μ be befined by* (6). *Than a system $\{f_j\}_1^\mu$ generates a cyclic subspace in $L_p^s[0,1]$ for J_s^α if and only if* $\det W_\mu\{f_1, \ldots, f_\mu\}(0) \neq 0$.

Definition 4 ([NV]). Let $A \in [X]$. Then

$$\operatorname{disc} A := \sup_{E \in \operatorname{Cyc} A} \min\{\dim E' \colon E' \subset E, E \in \operatorname{Cyc} A\}.$$

$\operatorname{disc} A$ is called a disc-characteristic of an operator A.

It is clear, that $\operatorname{disc} A \geq \mu_A$. It is important to note that $\operatorname{disc} A$ as well as μ_A depends not of A itself, but only on $\operatorname{Lat} A$.

Proposition 4. Let J_s^α be as above. Than

$$\operatorname{disc} J_s^\alpha = \mu_{J_s^\alpha}.$$

4. Commutant $\{J_s^\alpha\}'$ and the lattice $\operatorname{Hyplat} J_s^\alpha$

As usual, $\{T\}'$ stands for the commutant of the operator T defined on the Banach space X: $\{T\}' = \{R \in [X] \colon RT = TR\}$.

Recall that an invariant subspace $E \subset X$ of an operator $T \in [X]$ is called hyperinvariant for T if E is invariant for any bounded operator R that commutes with T, that is for $R \in \{T\}'$.

As usual, $\operatorname{Hyplat} T$ stands for the lattice of all hyperinvariant subspaces of T.

Proposition 5. Let $\operatorname{Re} \alpha > 0$ and $J_{s,0}^\alpha$ be the operator J^α defined on $L_{p,0}^s[0,1]$. Then $R \in \{J_{s,0}^\alpha\}'$ if and only if $R \in [L_{p,0}^s[0,1]]$ and

$$(Rf)(x) = \frac{d}{dx} \int_0^x r(x-t) f(t)\, dt, \qquad r(x) \in L_{p'}[0,1], \quad (p')^{-1} + p^{-1} = 1.$$

Corollary 4. *The lattices of the invariant and hyperinvariant subspaces of the operator $J_{s,0}^\alpha$ coinside:*

$$\operatorname{Hyplat} J_{s,0}^\alpha = \operatorname{Lat} J_{s,0}^\alpha = \{E_a^s \colon 0 \leq a \leq 1\}. \qquad E_a^s = \{f \in L_{p,0}^s[0,1] \colon f(x) = 0, x \in [0,a]\}.$$

Theorem 3. *Let either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\operatorname{Re} \alpha > k - \frac{1}{p}$ $(k-1 < s \leq k)$ and J_s^α be the operator J^α on $L_p^s[0,1]$. Then $R \in \{J_s^\alpha\}'$ if and only if*

$$(Rf)(x) = \frac{d}{dx} \int_0^x r(x-t) f(t)\, dt, \qquad r(x) \in W_p^k[0,1],$$

In particular. $\{J_s^\alpha\}'$ is commutative algebra and does not depend on α.

Corollary 5. *The double commutant $\{J_s^\alpha\}''$ of the operator J_s^α coinsides wih it's commutant:* $\{J_s^\alpha\}'' = \{J_s^\alpha\}' = \{J_s\}' = \{J_s\}''$.

Proposition 6. The lattice $\operatorname{Hyplat} J_s^\alpha$ is unicellular. $\operatorname{Hyplat} J_s^\alpha = \operatorname{Lat} J_k$.

Corollary 6. *Let $\alpha \neq 1$. Then* $\operatorname{Hyplat} J_s^\alpha = \operatorname{Lat} J_s^\alpha$ *if and only if* $0 \leq s \leq 1$.

Corollary 7. *Let $\alpha \in \mathbb{Z}_+ \setminus \{0\}$, $k-1 < s \le k$, $\alpha \le k$. Then*

$$\operatorname{Hyplat}^d J_s^\alpha = \pi^{-1}(\operatorname{Hyplat} J^\alpha(0;k))$$

if and only if k is odd and $\alpha = 2$.

Example 1. Let $\alpha = 2$ and $J_{3,5}^2\colon L_p^{3,5}[0,1] \to L_p^{3,5}[0,1]$. Then $\operatorname{Hyplat}^d J_{3,5}^2 = \operatorname{Lat}^d J_{3,5}^2 = \{E_0^{3,5}, E^{3,5}, E_2^{3,5}, E_3^{3,5}, E_4^{3,5}\}$, but $\pi^{-1}(\operatorname{Hyplat} J^2(0;3,5)) = \{E_0^{3,5}, E_2^{3,5}, E_4^{3,5}\}$.

Theorem 3 allows us to present a description of the algebra $\operatorname{Alg} J_s^\alpha$.

Proposition 7. The following are true:

1) If either $\alpha = 1$ or $s \le 1$, then $\operatorname{Alg} J_s^\alpha = \{J_s^\alpha\}''$;

2) If $1 < \alpha \le k-1 < s \le k$, then $\operatorname{Alg} J_s^\alpha = \{T = cI + R\colon c \in \mathbb{C}, R \in \operatorname{Alg}_0 J_s^\alpha\}$, where

$$\operatorname{Alg}_0 J_s^\alpha = \{R\colon Rf = r * f, r \in W_p^{k-1}[0,1], r^{(j)}(0) = 0 \quad \text{for } j \ne i\alpha - 1, \quad i \le \left[\frac{k-1}{\alpha}\right]\};$$

3) If $s \ge 2$ and $\operatorname{Re}\alpha \ge k - \frac{1}{p}$, then

$$\operatorname{Alg} J_s^\alpha = \{T = cI + R\colon c \in \mathbb{C}, Rf = r * f, r \in W_{p,0}^{k-1}[0,1]\};$$

Corollary 8. *$T \in \operatorname{Alg} J_s^\alpha$ if and only if the quotient operator*

$$\hat{T} \in \operatorname{Alg}(J(0;s)^\alpha) = \{J(0;s)^\alpha\}''.$$

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