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ON THE SPECTRAL THEORY OF OPERATOR MEASURES

1. Introduction Operator measures naturally arise in different questions of the spectral theory of selfadjoint operators (with spectrum of finite and infinite multiplicity), integral representations of operator-valued functions of Herglotz and Nevanlinna classes, in the theory of models for symmetric operators, etc.

Everywhere in the note H is a separable Hilbert space, $\Sigma(t) = \Sigma(t)^*$ is a nondecreasing strongly continuous from the left ($\Sigma(t-0) = \Sigma(t)$) operator-function on $\mathbb{R}$ with values in the $B(H)$. By a standart procedure (see [3], [4]) the function $\Sigma(t)$ generates an operator measure Σ, defined on the algebra $\mathcal{B}_b(\mathbb{R})$ of bounded Borel subsets of $\mathbb{R}$.

The theory of orthogonal measures (resolutions of the identity) is constructed in details. In this note we consider several questions for the theory of nonorthogonal operator measures. The principal role in our considerations is played by the Berezanskii-Gelfand-Kostyuchenko theorem (BGK-theorem) on differentiating of an operator measure.

We obtain an inner description of the space $L_2(\Sigma, H)$. Эта This problem has been posed by M. G. Krein [9] and in the case dim$H < \infty$ solved by I. S. Kac [1], [7], [8].

Further, we construct the theory of Hellinger spectral types for a nonorthogonal operator measure. We establish the existence of subspaces realizing Hellinger spectral types and in particular the existence of vectors of maximal type.

Some fact are new even for orthogonal measures. We show how the spectral Hellinger types of an operator $A = A^*$ can be found via a cyclic subspace L. It turnes out that the set of the vectors of maximal type, lying in L is an everywhere dense G_δ of second category.

Moreover an analog of the Zhordan Theorem for Operator measures-charges is established. For the simplicity, we state all the results for measures of the line, though they remain valid for measures on $\mathbb{R}^n$.

2. The space $L_2(\Sigma, H)$. Recall, following [3], the definition of the space $L_2(\Sigma, H)$. Let $C_{00}(H)$ be the set of all strongly continuous vector-function with finite support, and with values on a finite-dimensional subspace of H, depending on f. Further, for $f, g \in C_{00}(H)$ we define $(f,g)_{L_2(\Sigma,H)} = \int_{\mathbb{R}}(d\Sigma(t)f(t), g(t))_H$. (the intergal is understood as the limit of the Riemann sums). Faktorizing $C_{00}(H)$ by the lineal $L_0 = \{f : (f,f)_{L_2(\Sigma,H)} = 0\}$ and completing it we arrive at the Hilbert space $L_2(\Sigma, H)$.

Let $\mathfrak{S}_2(H)$ be the ideal of Hilbert-Schmidt operators in H and let $T \in \mathfrak{S}_2(H)$, ker$T$ =ker$T^* =$ $\{0\}$. Let also ρ be a scalar measure, equivalent to Σ ($\Sigma \sim \rho$).

By the BGK-Theorem the operator measure $\Sigma_T(\Delta) := T^*\Sigma(\Delta)T$ is differentiable in the weak sence with respect to ρ and its density $\Psi(t) := d\Sigma_T/d\rho (\geq 0)$ exists ρ-a.e. and takes values in $\mathfrak{S}_1(H)$.

Following [5] we can show that the derivative Ψ exists ρ-a.e. with respect to the $\mathfrak{S}_1(H)$-norm. (in [5] it is shown for an orthogonal measure $\Sigma = E$.)

Let $\widetilde{\mathfrak{H}}_t$ be the completion of $\mathcal{D}(T^{-1})$ with respect to the semi-norm

$$\|f\|^2_{\widetilde{\mathfrak{H}}_t} = (\Psi(t)T^{-1}f, T^{-1}f) = (\Psi(t)^{1/2}T^{-1}f, \Psi(t)^{1/2}T^{-1}f) \qquad (1)$$

Denote by $\mathfrak{H}_t$ the corresponding quotient space.

Theorem 1. *Let $T \in \mathfrak{S}_2(H)$ with $\ker T = \ker T^* = \{0\}$ and ρ be a scalar measure equivalent $\Sigma (\rho \sim \Sigma)$. Then the space $L_2(\Sigma, H)$ isometrically coincides with the direct intergal of the spaces $\mathfrak{H}_t$ by measure $\rho(t)$:*

$$L_2(\Sigma, H) = \int_{\mathbb{R}} \oplus \mathfrak{H}_t d\rho(t) =: \mathfrak{H}. \tag{2}$$

Moreover, the identity

$$\|f\|^2_{L_2(\Sigma,H)} = \int_{\mathbb{R}} \|\Psi(t)^{1/2} T^{-1} f(t)\|^2 d\rho(t) \tag{3}$$

holds for the dense in $L_2(\Sigma, H)$ set of vector-functions $f(t)$ with values in $H_+ = \mathcal{D}(T^{-1})$.

In particular, the space $\mathfrak{H}$ does not depend on the choice of T.

If $\Sigma(t) = E(t)$ is a resolution of the identity in H and $f(t) = E(\Delta)h$ with $h \in H_+$ and $\Delta \in \mathcal{B}_b(\mathbb{R})$ then identity (3) takes the form

$$(E(\Delta)h, h) \; (= \|f(t)\|^2_{L_2(\Sigma,H)}) = \int_{\Delta} \|\Psi(t)^{1/2} T^{-1} h\|^2_H d\rho(t).$$

This identity is equivalent to the direct integral form of the BGK Theorem for the orthogonal measure E.

In the case $\dim H < \infty$ Theorem 1 is equivalent to the I. Kac theorem [8], though our proof is much simplier than all the known ones. Observe, that there is a principal difference between the cases $\dim H < \infty$ and $\dim H = \infty$.

While for $\dim H < \infty$ the space $L_2(\Sigma, H)$ turnes out to be a space of ρ-measurabel vector-functions with values in H, this fails to be true for $\dim H = \infty$ even in the simplest cases.

Let for example $\Sigma_0 \geq 0$ be a compact operator in H and $\Sigma(t) = 0$ for $t \leq t_0$ and $\Sigma(t) = \Sigma_0$ for $t > t_0$. Thena $L_2(\Sigma, H) = H_-$ with H_- being the completion of H with respect to the negative norm $\|f\|_- = \|\Sigma_0^{1/2} f\|$.

3. The multiplicity function of a measure Σ.

Definition 1. Let $\rho \sim \Sigma$ and $\{e_i\}_{i=1}^{\infty}$ an orthonormal basis of H. Let further $\sigma_{ij}(t) := (\Sigma(t)e_i, e_j)$, $\psi_{ij}(t) := d\sigma_{ij}(t)/d\rho$ and $\Psi_n(t) := (\psi_{ij}(t))_{i,j=1}^n$.

Define the multiplicity function N_Σ and the general multiplicity $m(\Sigma)$ of an operator measure Σ, by letting

$$N_\Sigma(t) := \sup_{n \geq 1} \operatorname{rank} \Psi_n(t), \qquad m(\Sigma) := vraisup N_\Sigma(t) (\operatorname{mod} \rho). \tag{4}$$

The multiplicity function N_Σ is defined ρ-a.e. and it can be shown, that it is independent of the choice of a basis $\{e_i\}_1^{\infty}$.

Definition 2. a) Let Σ_1 and Σ_2 be operator measures on $\mathbb{R}$.

A measure Σ_1 is said to be subordinated to Σ_2 ($\Sigma_1 \prec \Sigma_2$), if Σ_1 is absolutely continuous with respect to Σ_2, that is $\Sigma_1(\delta) = 0$ as soon as $\Sigma_2(\delta) = 0$.

b) We say, that Σ_1 is spectrally subordinated to Σ_2 ($\Sigma_1 \prec\prec \Sigma_2$), if

$$\Sigma_1 \prec \Sigma_2 \quad \text{and} \quad N_{\Sigma_1} \leq N_{\Sigma_2} \; (\operatorname{mod} \Sigma_2).$$

The measures Σ_1 and Σ_2 are said to be spectrally equivalent if $\Sigma_1 \prec\prec \Sigma_2$ and $\Sigma_2 \prec\prec \Sigma_1$.

Let A be a selfadjoint operator in H, $E(t) := E_A(t)$ its resolution of the identity and L a subspace of H. Denote H_L the minimal A-invariant subspace, containing L :

$$H_L = \operatorname{span}\{E(\delta)L : \delta \in \mathcal{B}(\mathbb{R})\}.$$

A subspace L is cyclic ($L \in \operatorname{Cyc}(A)$) if $H_L = H$. For $L = \{\lambda g : = \lambda \in \mathbb{C}\}$, we set $H_g := H_L$.

The following theorem is proved with the help of Theorem 1.

Theorem 2. *Let Σ be a generalized resolution of the identity in $\mathcal{H}$, that is $\Sigma(-\infty) = 0$, $\Sigma(+\infty) = I_{\mathcal{H}}$. Let A be a selfadjoint operator in H and $E(t)$ its resolution of the identity. Then:*

a) $N_E(t)$ from Definition 1 coincides with the classical multiplicity function of E in the sence of [4], [11];

b) $\Sigma \prec\prec E$ if and only if there exist a Hilbert space $\widetilde{H} \supset \mathcal{H}$ and a unitary operator $U : H \to \widetilde{H}$ such that $\Sigma(t) = P_{\mathcal{H}} U E(t) U^* \lceil \mathcal{H}$ ($P_{\mathcal{H}}$ is the orthoprojection in $\widetilde{H}$ onto $\mathcal{H}$).

c) If $U^*\mathcal{H} \in \mathrm{Cyc}\, A$ then Σ is spectrally equivalent to E.

Conversely, if Σ is spectrally euivalent to E and $N_E(t)$ is E-a.e. finite (for example, if $m(E) < \infty$), then $U^*\mathcal{H} \in \mathrm{Cyc}\, A$.

In particular (for $H = \widetilde{H}$ and $U = I$) the measure Σ and its minimal orthogonal dilation are spectrally equivalent.

d) The resolution of the identity E_Q of the operator $Q : f \to xf$ in $L_2(\Sigma, H)$ is a minimal orthogonal dilation of Σ.

Theorem 2 complements the known Najmark theorema([1]), providing an answer to the question, which resolution of the identity can be a dilation of Σ.

Corollary 1. *The multiplication operators $Q_i : f \to xf$ in the spaces $L_2(\Sigma_i, H_i)$ $(i = 1, 2)$ are unitary equivalent iff Σ_1 and Σ_2 are spectrally equivalent.*

4. The elements of maximal type. Every operator measure Σ in H generates a family of σ-finite scalar measures μ_f $(\mu_f(\delta) := (\Sigma(\delta)f, f))$, defined on the σ-algebra $\mathcal{B}(\mathbb{R})$. It is clear that $\mu_f \prec \Sigma$ for all $f \in H$. It is known ([1], [4]) that any orthogonal measure E in H posesses an element f of maximal type, that is such, that $\mu_f \sim E$.

It turnes out, that this fact also holds true for nonorthogonal measures. Moreover, the following stronger result is valid. We note, that this result is new even for orthogonal measures.

Theorem 3. *Let $\Omega_\Sigma := \{f \in H : \Sigma \sim \mu_f\}$ be the set of all vectors of maximal type for an operator measure Σ in H. Then:*
a) Ω_Σ is an everywhere dense G_δ-set of second category in H;
b) $\omega(\Omega_\Sigma) = 1$ for any Gaussian measure ω in H.

Corollary 2. *Let A be a selfadjoint operator in H H, $E(t)$ its resolution of the identity and $L \in \mathrm{Cyc}(A)$. Then:*

a) Ω_E is an everywhere dense G_δ-set of second category in L;

b) $\omega(\Omega_E \cap L) = 1$ for any Gaussian measure ω in L.

5. The Hellinger types. The class of all Borel measures, equivalent to a measure μ is called the type of the measure μ and is denoted by $[\mu]$ (see [4]. Let $A = A^*$, $E := E_A$, $g \in H$ and $\mu_g : \delta \to \mu_g(\delta) := (E(\delta)g, g)$, $\delta \in \mathcal{B}(\mathbb{R})$. The type $[g]$ of an element $[g]$ (with respect to E) is the type of the measure μ_g, $[g] = [\mu_g]$.

Consider an orthogonal decomposition of the form $H = \oplus_{i=1}^{m} H_{g_i}$, $(m \leq \infty)$ If the types of the elements g_i do not increase, $[g_{i+1}] \prec [g_i]$, then their number $m(\leq \infty)$ and types are defined uniquely and referred to as the Hellinger types of the measure E. They form (see [4]) a complete set of unitary invariants of the operator A.

Let $g_1 \in \Omega_E := \{g \in H : \ \mu_g \sim E\}$. Then $\mu_g \prec \mu_{g_1} := \mu$ for all $\in h$ and the type $[g]$ is uniquely determined by the support of the measure μ_g with respect to $\mu : \Gamma(g) := \{t \in \mathbb{R} : d\mu_g/d\mu > 0\}$. Therefore the Hellinger types are uniquely determined (mod E) by their supports $\Gamma_i(E) := \Gamma(g_i). i \leq m$.

The sets $\Gamma_i(E)$ themselves are determined (see [4]) by the multiplicity function (and the measure μ): $\Gamma_i(E) = \{t \in \mathbb{R} : N_E(t) \geq i\}$.

The existence of the multiplicity function N_Σ of the form (3) allows us to introduce the i-th Hellinger type for a nonorthogonal measure Σ as the type of the scalar measure $d\mu_i := \chi_i d\rho$ with $\rho \sim \Sigma$ and χ_i being the indicator of the set

$$\Gamma_i(\Sigma) = \{t \in \mathbb{R} : N_\Sigma(t) \geq i\}, \qquad i \in \{1, \dots, m(\Sigma)\}. \tag{5}$$

Let us call $\Gamma_i(\Sigma)$ the support of the i-th Hellinger type of the measure Σ. It is clear that $\Gamma_i(\Sigma) \supset \Gamma_{i+1}(\Sigma)$. Let i_0 be the number of Γ_i, equivalent to $\Gamma_1(\Sigma)$ (mod ρ), that is $\rho(\Gamma_1(\Sigma) \setminus \Gamma_i(\Sigma)) = 0$.

If $g_1 \in \Omega_\Sigma$ (by Theorem 3 $\Omega_\Sigma \neq \emptyset$,) then $\mu_g \prec \mu_{g_1} =: \mu \sim \Sigma$ with $\mu_g(\delta) := (\Sigma(\delta)g, g)$. Therefore the set $\Gamma(g) := \{t : d\mu_g/d\mu > 0\}$ is a (nontopological) support of the measure μ_g (i.e. $\mu_g(\mathbb{R} \setminus \Gamma(g)) = 0$).

It turnes out that, though $\Omega_\Sigma \neq \emptyset$, elements of "junior" types (that is vectors $g \in H \setminus \Omega_\Sigma$, such that $\Gamma(g) = \Gamma_i(\Sigma)$ for some $i > i_0$) may happen not to exist.

Example 2. Let for example

$$\Sigma(t) = \sum_{i<t} P_i$$

be a 2×2 discrete measure with jumps

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the points $1, 2, 3$. It is easy to see that $N_\Sigma(1) = N_\Sigma(2) = 1$ and $N_\Sigma(3) = 2$. Therefore $\Gamma_1(\Sigma) = \{1, 2, 3\}, \Gamma_2(\Sigma) = \{3\}$. If $h = (h_1, h_2)$, then $\Gamma(h) = \{1, 2, 3\} = \Gamma_1(\Sigma)$ for $h_1 h_2 \neq 0$. Further, $\Gamma(h) = \{2, 3\}$, if $h_1 = 0$ and $\Gamma(h) = \{1, 3\}$ if $h_2 = 0$. Therefore $\Gamma(h) \neq \{3\} = \Gamma_2(\Sigma)$ for any h.

The wish to realize the "junior" Hellinger types with the help of some subspaces forced us to introduce the following

Definition 3. A subspace $L = L_k$ ($\dim L_k = k$) is called a k-th Hellinger subspace for an operator measure Σ in H if

$$\Gamma_i(P_L \Sigma \lceil L) = \Gamma_i(\Sigma) \quad \text{for all} \quad i \leq k,$$

where P_L is the orthoprojection onto L.

In particular, a vector of maximal type generates a one-dimensional (first) Hellinger subspace.

Theorem 4. *Under the hypothesis of Theorem 3 for each $h \in \Omega_\Sigma$ there exists a chain of Hellinger subspaces (Hellinger chain):*

$$\{\lambda h\} =: H_1 \subset H_2 \subset \dots \subset H_k \subset \dots \subset H_m, \quad \dim H_k = k. \tag{6}$$

Let $H_1 \subset \dots \subset H_m$ be a chain of subspaces of the form (6) and $\{e_i\}_1^m$ a basis in в H_m, such that $H_k = \text{span}\{e_i\}_1^k$, $k \in \{1, \dots, m\}$. Let us set $\sigma_{ij}(t) := (\Sigma(t)e_i, e_j)$, $\psi_{ij}(t) := d\sigma_{ij}(t)/d\rho$ and $\Psi_k(t) := (\psi_{ij}(t))_{i.j=1}^k$. Then the chain $H_1 \subset \dots \subset H_m$ is a Hellinger chain iff

$$\Gamma_k(\Sigma) = \{t \in \mathbb{R} : \det \Psi_k(t) \neq 0\} \; (\text{mod}\, \Sigma), \quad k \in \{1, \dots, m\}. \tag{7}$$

Thus, Theorem 4 amounts to saying that there exists an orthonormal system $\{e_i\}_1^m$ in H, such that the k-th Hellinger type is realized by the measure $(\wedge^k \Psi(t)\varphi_k, \varphi_k)d\mu$, where $\wedge^k \Psi(t)$ is the k-th exterior power of $\Psi(t) := d\Sigma_T/d\mu$, and $\varphi_k := e_1 \wedge \wedge e_k$ is a k-vector, $\varphi_k \in \wedge^k(H)$.

Corollary 3. *Let A be a selfadjoint operator in H and $E(t)$ its resolution of the identity, $L \in \text{Cyc}(A)$ and $h \in \Omega_E \cap L$.. Then there exists a chain of Hellinger subspaces in L of the form* (6), *$H_k \in L$, $k \in \{1, \dots, m\}$.*

If the multiplicity $m = m(E)$ of the measure E is finite, then $H_m \in \text{Cyc}\, A$.

If the vectors $\{e_i\}_{i=1}^m$ are spectrally orthogonal with respect to E and H_k :=span$\{e_i\}_1^k$, then the chain $H_1 \subset \ldots \subset H_m$ is a Hellinger chain if and only if $\Gamma(e_i) = \Gamma_i(E)$, $i \le m$.

If $L \in \mathrm{Cyc}(A)$, a system of spectrally orthogonal vectors, realizing Helliger types does not exist in general. But, according to Corollary 3 these types are realized by means of subspaces $H_k \subset L$ or, equivalently, the k-vectors $\varphi_k = e_1 \wedge \ldots \wedge e_k \in \wedge^k(L)$.

6. An analog of the Zhordan Theorem. The results of this section have obtained by the second author only.

Definition 4. An operator-function $\Sigma = \Sigma^* : \mathcal{B}(\mathbb{R}) \to B(H)$ (operator measure-charge) is of weakly bounded variation on $\mathbb{R}$, if $\mu_{f,g} : \delta \to (\Sigma(\delta)f, g)$ is a finite charge on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ for any $f, g \in H$.

Theorem 5. *An operator measure-charge Σ of weakly bounded variation on $\mathbb{R}$ can be be expressed as the difference of two finite nonnegative operator measures $\Sigma = \Sigma_1 - \Sigma_2$ if and only if*

$$\mathrm{Var}_{\mathbb{R}} \|T^*\Sigma(\Delta)T\|_1 := \sup_{\pi} \sum_i \|T^*\Sigma(\Delta_i)T\|_1 =: c(T) < \infty$$

for any $T \in \mathfrak{S}_2(H)$, where $\|\cdot\|_1$ is the trace norm and the supremum is taken over all partitions $\pi = \{t_j\}_{-\infty}^{\infty}$ of $\mathbb{R}$, $\Delta_i = [t_i, t_{i+1})$.

The proof is based on some facts from the theory of C^*-algebras and completely bounded maps [12].

Let $x_i^{(n)} = x_i^{(n)*}$ be $2^n \times 2^n$-Clifford matrices, that is $x_i^{(n)}x_j^{(n)} + x_j^{(n)}x_i^{(n)} = 2\delta_{ij}I$, $\quad i, j \in \{1, \ldots, n\}$. Define the operator measure-charge

$$\Sigma(\Delta) = \oplus_1^{\infty}\Sigma_n(\Delta), \qquad \Sigma_n(\Delta) = 1/\sqrt{2n} \sum_{1/k\in\Delta,\ k\le n} x_k^{(n)}.$$

Clearly, $\Sigma(\Delta)$ is a discrete measure with the support $\{0\} \cup \{1/k\}_{k=1}^{\infty}$. It is of weakly bounded variation, but can not be expressed as the difference of two nonnegative operator measures.

The idea of use of the Clifford matrices is taken from [10], where it was used in a different context.

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