Ученые записки Таврического национального университета им. В. И. Вернадского
Серия "Математика. Механика. Информатика и Кибернетика"N 1 (2002) 142 – 147
УДК 517.95

V. V. LIMANSKY, D. V. LIMANSKY

ON DIMENSION OF A SYSTEM OF DIFFERENTIAL OPERATORS WITHOUT MIXED DERIVATIVES

The example of a system of maximal linearly independent differential polynomials, consisting of "pure"derivatives and having the space of subordinated operators of intermediate dimension is given.

Keywords: Dirichlet boundary-value problem, harmonic function, Stone-Weierstrass theorem

1. INTRODUCTION

In this paper we study conditions for a differential polynomial $Q(D)$ to be subordinate to a system of other ones $\{P_j(D)\}_1^N$ in the spaces $L_p(\Omega)$. In other words, we consider the problem of description of the linear spaces $L_\Omega(P_1, \dots, P_N)$, depending, in general, on Ω and $p \in [1;\infty]$, of differential operators $Q(D)$, satisfying the estimate

$$\|Q(D)f\|_{L_p(\Omega)} \leqslant C\left[\sum_{j=1}^{N} \|P_j(D)f\|_{L_p} + \|f\|_{L_p}\right], \quad \forall f \in C^\infty(\overline{\Omega}) \tag{1}$$

with some constant C independent of $f \in C^\infty(\overline{\Omega})$. Here $D = (D_1, \dots, D_n)$, $D_j = \partial/\partial x_j$.

The above problem was completely solved by Hörmander in the case $N = 1$ (and $p = 2$) [1] (see also [2], [3]). For $N > 1$ the spaces $L(P_1, \dots, P_N)$ were described in some cases. More precisely, the coercivity criterion, i.e. a criterion for the maximal possible dimension of the spaces $L(P_1, \dots, P_N)$ for $1 < p < \infty$, has been obtained by K.T.Smith [5] and O.V.Besov [6] (see also [4]). Further, the spaces $L(P_1, \dots, P_N)$ were described by V.P.Il'in for a number of domains Ω ($\subset \mathbb{R}^n$) in the case of differential monomials $P_j(D)$ (see [4], p. 13). And, finally, two classes of operators $\{P_j(D)\}_1^N$ for which the dimension of the space $L(P_1, \dots, P_N)$ is minimal possible, i.e. $\dim L(P_1, \dots, P_N) = N + 1$, were indicated by M.M.Malamud in [7].

For further considerations we need the following result from [7].

Theorem 1. [7] *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\{P_j(D)\}_1^N$ be differential operators with constant coefficients. If*

a) their symbols $\{P_j(\xi)\}_1^N$ are algebraically independent;

b) the generic fiber of the mapping $P = (P_1, \dots, P_N) : \mathbb{C}^n \to \mathbb{C}^N$ is irreducible (i.e. the algebraic manifolds

$$P^{-1}(\alpha) = V_\alpha = V(P_1 - \alpha_1, \dots, P_N - \alpha_N) = \{x \in \mathbb{C}^n : P_1(x) - \alpha_1 = \ldots = P_N(x) - \alpha_N = 0\}$$

are irreducible for almost all $\alpha \in \mathbb{C}^N$), then estimate (1) *is equivalent to the equality*

$$Q(D) = \lambda_0 + \sum_{j=1}^{N} \lambda_j P_j(D) \tag{2}$$

with some $\lambda_j \in \mathbb{C}$, $0 \leqslant j \leqslant N$, i.e. to the equality $\dim L(P_1, \dots, P_N) = N + 1$.

As a corollary of this result, it was shown in [7] that for linearly independent operators

$$P_j(D) = \sum_{k=1}^{n} a_{jk} D_k^{l_k}, \quad l_k > 0, \quad 1 \leqslant k \leqslant n; \quad 1 \leqslant j \leqslant N = n - 1, \tag{3}$$

whose linear span contains no differential monomials $D_k^{l_k}$, $k \leqslant n$, the equality $\dim L(P_1, \dots, P_N) = N + 1$ is true.

In [7] M. M. Malamud formulated the following conjecture. Does the equality $\dim L(P_1, \dots, P_N) = N + 1$ hold true for a system of linearly independent operators

$$P_j(D) = \sum\nolimits_{k=1}^{n} a_{jk} D_k^{l_{jk}}, \quad l_{jk} > 0, \quad 1 \leqslant k \leqslant n; \quad 1 \leqslant j \leqslant N = n - 1 \tag{4}$$

with an arbitrary matrix of degrees (l_{jk}) if span $\{P_j(D)\}_1^N$ contains no polynomials in one variable? In [7] the validity of this hypothesis was established in two cases:

1) a matrix of coefficients (a_{jk}) is triangular, i. e. $a_{jk} = 0$ for $j > k$;

2) a matrix of exponents (l_{jk}) is of the following form: $l_{jk} = l_k$, $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant N$.

In the case $N = 2$ the conjecture was set in [10]. The condition of algebraic nature (Theorem 1) under which the equality $\dim L(P_1, \dots, P_N) = N + 1$ holds true, was formulated and proved in [9]. Using this statement we have confirmed the hypothesis of M. M. Malamud for $N = 2$ in more general formulation. Some particular cases of the hypothesis have been established in [8] for $N = 3$.

If $N > 2$, the conjecture was proved in a modified form.

Theorem 2. [11] *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\{P_j(D)\}_1^N$ be differential polynomials without mixed derivatives:*

$$P_i(D) = \sum\nolimits_{j=1}^{n} P_{ij}(D_j), \quad P_{ij} \in \mathbb{C}[z_k], \quad P_{ij} \neq const, \quad n > N. \tag{5}$$

Suppose also that:

a) their symbols $\{P_j(\xi)\}_1^N$ are linearly independent;

b) span $\{P_j(D)\}_1^N$ does not contain k linearly independent polynomials in k variables for each $k \leqslant N - 1$.

Then estimate (1) *is equivalent to equality* (2) *with some $\lambda_j \in \mathbb{C}$, $0 \leqslant j \leqslant N$, i.e. to the equality* $\dim L(P_1 \dots, P_N) = N + 1$.

In this communication we present the following result.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^4$ defined below, and let $\{P_j(D)\}_1^4$ be the following differential polynomials in $L_2(\Omega)$:*

$$P_1(D) := \sum\nolimits_1^4 D_j^4, \quad P_2(D) := D_1^2 - D_2^2, \quad P_3(D) := D_1^3 - D_2^3, \quad P_4(D) := D_1 - D_2. \tag{6}$$

Then $P_1 \in L(P_1, P_2, P_3)$, i.e., $\dim L(P_1, P_2, P_3) > 4$.

Remark. It is obvious that $P_4 \notin \operatorname{span}(P_1, P_2, P_3)$, so $\dim L(P_1, P_2, P_3) > 4$. Thus, Theorem 3 shows that M.M.Malamud's conjecture from [7] is not true.

Proving this Theorem is the main purpose of this work.

2. Sketch of the proof of Theorem 3

2.1. Change of domain. Changing the variables

$$x_1' = \sqrt{3}(x_1 - x_2), \quad x_2' = x_1 + x_2, \quad x_3' = x_3, \quad x_4' = x_4, \tag{7}$$

we obtain a new form of operators $\{P_j(D)\}_2^4$:

$$P_2(D') = 4\sqrt{3} D_1' D_2', \quad P_3(D') = 6\sqrt{3} D_1' ({D_1'}^2 + {D_2'}^2) := 6\sqrt{3} D_1' \Delta', \quad P_4(D') = 2\sqrt{3} D_1'.$$

Here D' denotes the operator D in the coordinates x_i'. The change of variables (7) takes the domain Ω to the cube $\Omega' := (0; \pi)^4$. Hence the including $P_4 \in L_\Omega(P_2, P_3)$ is equivalent to the including

$$D_1 \in L(D_1 D_2, D_1 \Delta) \tag{8}$$

in the space $L_2(\Omega)$, where $\Omega = (0;\pi)^4$. (The dashes over all symbols are omitted).

2.2. Converse assumption. Suppose that (8) is not true. This means the existing of functions $f_n \in C^\infty(\overline{\Omega})$ such that

$$f_n \to 0, \quad D_1D_2f_n \to 0, \quad D_1\Delta f_n \to 0 \text{ as } n \to \infty; \quad \|D_1 f_n\| = 1, \ \forall n \in \mathbb{N}. \qquad (9)$$

Here and in the sequel we mean the convergence and norms with respect to L_2 - topology (if the contrary is not mentioned).

2.3. Reduction to real-valued functions f_n. From the equality $\|D_1 f_n\| = 1$ we obtain that either $\|D_1 \operatorname{Re} f_{n_k}\| > \alpha$ or $\|D_1 \operatorname{Im} f_{n_k}\| > \alpha$ for some $\alpha > 0$ and some subsequence $\{n_k\}$. Then, substituting f_n for $\frac{\operatorname{Re} f_n}{\|D_1 \operatorname{Re} f_n\|}$ or $\frac{\operatorname{Im} f_n}{\|D_1 \operatorname{Im} f_n\|}$ respectively we can assume that the relations (9) are satisfied with real-valued functions $\{f_n\}_1^\infty$.

2.4. Reduction to harmonic functions $D_1 f_n$. Consider the following boundary-value problem in Ω:

$$\Delta u_n = D_1\Delta f_n, \qquad (10)$$

$$u_n|_{\Gamma\times K} = 0\,, \qquad (11)$$

where $K := [0;\pi]^2$ (with respect to x_3 and x_4), $\Gamma := \partial K$ and $\Delta := \sum_{i=1}^4 D_i^2$. Let u_n be a solution of the problem (10) - (11). Taking into account the apriori estimate

$$\|u_n\|_{W_2^2(\Omega)} \leqslant C\|D_1\Delta f_n\|_{L_2(\Omega)}$$

for the Dirichlet boundary-value problem (10) - (11), we obtain from (9) that $u_n \to 0$, $D_1 u_n \to 0$, $D_2 u_n \to 0$.

Put $w_n = w_n(x_1,\ldots,x_4) := \int_0^{x_1} u_n(t,x_2,x_3,x_4)\,dt$. Then the inequality $\left|\int_0^{x_1} u_n\,dt\right| \leqslant \sqrt{x_1}\|u_n\|$ implies that $w_n \to 0$. Hence we conclude that all relations (9) remain valid with f_n replaced by $\frac{f_n - w_n}{\|D_1(f_n-w_n)\|}$. Moreover, rewriting (10) in the form $\Delta(D_1(f_n - w_n)) = 0$ we arrive at the following relations:

$$f_n \to 0, \quad D_1D_2f_n \to 0 \text{ as } n \to \infty; \quad D_1\Delta f_n = 0, \quad \|D_1 f_n\| = 1, \ \forall n \in \mathbb{N}. \qquad (12)$$

In other words, we can assume $D_1 f_n$ to be harmonic functions in Ω.

2.5. Decomposition of $D_1 f_n$ in the sum of two functions. $D_1 f_n = h_n + g_n$, where $g_n = \int_0^{x_2} D_1D_2f_n(x_1,t,x_3,x_4)\,dt$, and h_n does not depend on x_2. Additionally,

$$g_n \to 0, \quad g_n|_{x_2=\pi} \to 0 \text{ as } n \to \infty; \quad g_n|_{x_2=0} = 0, \ \forall n \in \mathbb{N}.$$

2.6. Reduction to even and odd cases. Applying decompositions of the form

$$\varphi(x_1,\ldots,x_4) = \frac{\varphi(x_1,\ldots,x_4) + \varphi(x_1,\pi - x_2,x_3,x_4)}{2} + \frac{\varphi(x_1,\ldots,x_4) - \varphi(x_1,\pi - x_2,x_3,x_4)}{2},$$

we can assume that

$$f_n\big|_{x_2=x_2'} = f_n\big|_{x_2=\pi-x_2'}, \quad g_n\big|_{x_2=x_2'} = g_n\big|_{x_2=\pi-x_2'}, \quad \forall n \in \mathbb{N};$$
$$g_n \to 0, \quad g_n|_{x_2=0} = g_n|_{x_2=\pi} \to 0 \text{ as } n \to \infty.$$

Applying decompositions of the form

$$\varphi(x_1,\ldots,x_4) = \frac{\varphi(x_1,\ldots,x_4) + \varphi(\pi - x_1,x_2,x_3,x_4)}{2} + \frac{\varphi(x_1,\ldots,x_4) - \varphi(\pi - x_1,x_2,x_3,x_4)}{2},$$

we can reduce our conditions to two cases:

1. Odd case:

$$f_n\big|_{x_1=x_1'} = -f_n\big|_{x_1=\pi-x_1'},\; g_n\big|_{x_1=x_1'} = g_n\big|_{x_1=\pi-x_1'},\; h_n\big|_{x_1=x_1'} = h_n\big|_{x_1=\pi-x_1'},\quad \forall n\in\mathbb{N}. \tag{13}$$

2. Even case:

$$f_n\big|_{x_1=x_1'} = f_n\big|_{x_1=\pi-x_1'},\; g_n\big|_{x_1=x_1'} = -g_n\big|_{x_1=\pi-x_1'},\; h_n\big|_{x_1=x_1'} = -h_n\big|_{x_1=\pi-x_1'},\quad \forall n\in\mathbb{N}. \tag{14}$$

2.7. **Approximation of h_n with trigonometric polynomials.** First we consider the odd case (13). Let $\|g_n\|<\varepsilon$, $\|g_n|_{x_2=0}\|<\varepsilon$. Then $D_1f_n=\theta h_n+(h_n-\theta h_n)+g_n$. Here we define a function θ such that

$$\theta=\theta(x_1):=\begin{cases}0, & x_1\in(-\infty;0]\cup[\pi;+\infty);\\ 1, & x_1\in[\alpha;\pi-\alpha],\end{cases}$$

and

$$0\leqslant\theta(x_1)\leqslant 1,\quad \theta(x_1)\in C^\infty(\mathbb{R}),\quad \alpha=\frac{\varepsilon^2}{2C^2\pi^3},\quad |h_n|<C,\quad \|(h_n-\theta h_n)+g_n\|<2\varepsilon.$$

By Stone – Weierstrass theorem, $\left|\theta h_n(x_1,x_3,x_4)-\sum_{s=1}^N a_s(x_3,x_4)\sin sx_1\right|<\frac{\varepsilon}{\pi^2}$. It follows that

$$D_1f_n=\sum_{s=1}^N a_s\sin sx_1+p_n,\quad \|p_n\|<3\varepsilon.$$

2.8. **Estimate for the Fourier coefficients.** Since $a_s=\frac{2}{\pi}\int_0^\pi\left(\sum_1^N a_k\sin kx_1\right)\sin sx_1\,dx_1=\frac{2}{\pi}\int_0^\pi(D_1f_n-p_n)\sin sx_1\,dx_1=-\frac{2}{\pi}\left(\int_0^\pi p_n\sin sx_1\,dx_1+\int_0^\pi sf_n\sin sx_1\,dx_1\right)$, we have

$$\|a_s\|\leqslant\frac{2}{\sqrt{\pi}}\varepsilon\sqrt{9+s^2}.$$

Consequently, $D_1f_n=\sum_{k=T+1}^N a_k\sin kx_1+q_n$, $\|q_n\|\leqslant R_1(T)\varepsilon$. (Here T is some constant that will be choosen later).

2.9. **Solution of the Dirichlet problem.** Using functions of the type θ and Stone – Weierstrass theorem once more, we represent the restriction of D_1f_n to $\Gamma\times K$ as a sum of three functions $\gamma_1+\gamma_2+\gamma_3$, where $\gamma_1:=\sum_{T+1}^N a_k\sin kx_1$, $\gamma_2:=\sum_1^M b_l\sin lx_2$, and γ_3 satisfies $\|\gamma_3\|<R_2(T)\varepsilon$. Therefore,

$$D_1f_n=\sum_{k=T+1}^N a_k\sin kx_1\frac{\operatorname{ch}k(x_2-\pi/2)}{\operatorname{ch}(k\pi/2)}+\sum_{l=1}^M b_l\sin lx_2\frac{\operatorname{ch}l(x_1-\pi/2)}{\operatorname{ch}(l\pi/2)}+\sigma_n(x),$$

where $\|\sigma_n(x)\|<R_3(T)\varepsilon$. It follows that $\varphi_n+\psi_n+\mu_n=0$, where

$$\varphi_n:=\sum_{k=T+1}^N a_k\sin kx_1\left(\frac{\operatorname{ch}k(x_2-\pi/2)}{\operatorname{ch}(k\pi/2)}-1\right),\quad \psi_n:=\sum_{l=1}^M b_l\sin lx_2\frac{\operatorname{ch}l(x_1-\pi/2)}{\operatorname{ch}(l\pi/2)},$$

$$\|\mu_n\|<R_4(T)\varepsilon.$$

2.10. Estimates for $\|\varphi_n\|$, $\|\psi_n\|$ and (φ_n, ψ_n). Choice of T. After some computations we obtain

$$\|\varphi_n\|^2 = \sum\nolimits_{k=T+1}^{N} \alpha_k \|a_k\|^2, \quad \|\psi_n\|^2 = \sum\nolimits_{l=1}^{M} \beta_l \|b_l\|^2, \quad (\varphi_n, \psi_n) = \sum\nolimits_{k=T+1}^{N} \sum\nolimits_{l=1}^{M} \gamma_{kl}(a_k, b_l),$$

and $\frac{\gamma_{kl}^2}{\alpha_k \beta_l} < \frac{16k^6}{l(k^2+l^2)^4}$. Cauchy – Schwarz – Bunyakovskii inequality gives

$$|(\varphi_n, \psi_n)| \leqslant \sqrt{\sum\nolimits_{k,l} \frac{\gamma_{kl}^2}{\alpha_k \beta_l}} \|\varphi_n\| \, \|\psi_n\|. \tag{15}$$

The series $\sum_{k,l} \frac{16k^6}{l(k^2+l^2)^4}$ is convergent. Hence we can choose T such that

$$\sum\nolimits_{k=T+1}^{N} \sum\nolimits_{l=1}^{M} \frac{\gamma_{kl}^2}{\alpha_k \beta_l} < \frac{1}{4}. \tag{16}$$

2.11. Contradiction with the assumption. Using (15) and (16), we get $|(\varphi_n, \psi_n)| < \frac{1}{2}\|\varphi_n\| \, \|\psi_n\|$. It follows that $\|\varphi_n + \psi_n\| \geqslant \frac{1}{\sqrt{2}}\|\varphi_n\|$ and $\frac{1}{\sqrt{2}}\|\varphi_n\| \leqslant \|\varphi_n + \psi_n\| = \|\mu_n\| < R_4(T)\varepsilon$, i.e., $\|\varphi_n\| < \sqrt{2}R_4(T)\varepsilon$. Further, we denote

$$\tau_n := \sum\nolimits_{k=T+1}^{N} a_k \sin kx_1 \frac{\operatorname{ch} k(x_2 - \pi/2)}{\operatorname{ch}(k\pi/2)}, \quad \kappa_n := \sum\nolimits_{k=T+1}^{N} a_k \sin kx_1.$$

It can be shown that $\|\tau_n\| \leqslant 0,8\|\kappa_n\|$ and, hence, $\|\varphi_n\| \geqslant \|\tau_n\| - \|\kappa_n\| \geqslant 0,2\|\tau_n\|$, $\|\tau_n\| < 5\sqrt{2}R_4(T)\varepsilon$; $\|\kappa_n\| = \|\varphi_n - \tau_n\| \leqslant \|\varphi_n\| + \|\tau_n\| < R_5(T)\varepsilon$. But $D_1 f_n = \kappa_n + q_n$. We have that

$$1 = \|D_1 f_n\| \leqslant R_5(T)\varepsilon + R_1(T)\varepsilon \to 0 \text{ as } n \to \infty. \tag{17}$$

The relation (17) gives a contradiction.

The even case (14) is considered similarly. Theorem 3 is proved.

References

[1] Lars Hörmander, *On the theory of general partial differential operators*. Acta Math. 94(1955), 161-248.

[2] Lars Hörmander, *The analysis of linear partial differential operators*. Vol.II, Springer-Verlag, Berlin, 1983.

[3] Yu.M. Berezanskiĭ, *Expansion in eigenfunctions of selfadjoint operators*, "Naukova Dumka", Kiev, 1965; English transl., Amer. Math. Soc., Providence, RI, 1968.

[4] O.V. Besov, V.P. Il'in, and S.M. Nikol'skiĭ, *Integral representations of functions and imbedding theorems*, "Nauka", Moscow, 1996.

[5] K.T. Smith, *Inequalities for formally positive integro-differential forms* // Bull. Amer. Math. Soc. **67** (1961), 368-370.

[6] O.V. Besov, *On coercivity in nonisotropic Sobolev spaces* // Math. Sb. **73 (115)** (1967), 585-599, English transl. in Math. USSR Sb. **2** (1967).

[7] M.M. Malamud, *Estimates for systems of minimal and maximal differential operators in $L_p(\Omega)$* // Trans. Moscow Math. Soc. **56** (1995), 206-261.

[8] D.V. Limansky, *On subordinated conditions for systems of differential operators in the spaces $L_p(\Omega)$. The case of three operators* // Works of the Institute of Applied Mathematics and Mechanics. Vol.4 — Donetsk, 1999. - pp. 100-106 (Russian).

[9] V.V. Limansky, D.V. Limansky, *On estimates for a system of maximal differential operators* // Methods of Functional Analysis and Topology. Vol.4 (1998), N°3, pp.45-57.

[10] V.V. Limansky, D.V. Limansky, *On estimates for one maximal differential operator to be subordinated to a system of two others* // Reports of National Academy of Sciences of Ukraine. N°12, 1998. - pp.24-28 (Russian).

[11] D.V. Limansky, *On estimates for a system of differential polynomials without mixed derivatives* // Spectral and Evolution Problems: Proceedings of the Eleventh Crimean Autumn Mathematical School-Simposium (KROMSH-XI). Vol. 11. Simferopol, 2001. – P. 117-120.

Поступила в редакцию 11.03.2002