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BANACH ALGEBRAS ASSOCIATED WITH AUTOMORPHISMS. STRUCTURAL PROPERTIES. OPERATORS WITH MEASURABLE COEFFICIENTS

In the present paper we continue the study of the structure of a Banach algebra $B(A,T_g)$ generated by a certain Banach algebra A of operators acting in a Banach space D and a group $\{T_g\}_{g\in G}$ of isometries of D such that $T_gAT_g^{-1} = A$. We investigate the interrelations between the existence of the expectation of $B(A,T_g)$ onto A, metrical freedom of the automorphisms of A induced by T_g and the dual action of the group G on $B(A,T_g)$. The results obtained are applied to the description of the structure of Banach algebras generated by 'weighted composition operators' acting in Lebesgue spaces.

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1. INTRODUCTION

This article should be considered as a 'measurable counterpart' of [1]. As in [1] the principal object under consideration here is a Banach algebra $B(A,T_g)$ generated by a certain Banach algebra A of operators acting in a Banach space D and a group $\{T_g\}_{g\in G}$ of isometries of D (a representation $g\to T_g$ of a discrete group G) such that

$$T_gAT_g^{-1} = A, \qquad g\in G \tag{1.1}$$

which means that T_g generates the automorphism $\hat{T}_g$ of A given by

$$\hat{T}_g(a) = T_gaT_g^{-1}, \qquad a\in A. \tag{1.2}$$

In [1] we obtained some principle characteristics of the structure of such algebras and also considered a number of examples where the role of A was played by algebras of continuous operator valued functions. In the present article we continue this investigation and present the results on the structure of the corresponding algebras in the situation when as A are taken algebras of measurable operator valued functions acting in Lebesgue spaces. Therefore roughly speaking the material of the paper describes a passage from the 'topological' picture given in [1] to a 'measurable' one.

We recall that in the Hilbert space situation (that is in the C^*–algebra theory) the analogous objects are closely related to the crossed products (see, for example [2]) and description of their structure is the theme of numerous investigations. In particular, Landstad [3] presented the necessary and sufficient conditions (in terms of *duality theory*) for a C^*–algebra to be isomorphic to a crossed product (of an algebra and a locally compact group of automorphisms). In the case of a discrete group in [4], Chapter 2 there were found the conditions for a C^*–algebra to be isomorphic to a crossed product in terms of the group action (the so called *topologically*

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free action (see 1.3)) and also in terms of satisfaction of a certain inequality (*property (*)* (1.3)) guaranteeing the existence of the expectation of the algebra $B(A, T_g)$ onto the algebra A (see (1.4), (1.5)).

In [1] we have shown that the mentioned properties (topologically free action, property (*) and dual action of the group) play an exceptional role in the general Banach space situation as well. By means of these properties there were established a number of results describing the structure of of $B(A, T_g)$ up to isomorphism.

In this paper we show (in Section 2) that in the 'measurable' situation the natural substitute for the topologically free action is the *metrically free* action (see 2.2). We investigate the interrelation between these notions and in particular find out that from a ceratin point of view they are equivalent. This enables us to transfer the main structural results of [1] from the 'topological' to the 'measurable' situation.

Since in the general Banach space situation we do not have a universal object like the crossed product in the Hilbert space situation to describe the structure of $B(A, T_g)$ we have to specify the algebra A and the isometries T_g. Sections 3-5 are devoted to the applications of the results obtained in Section 2 and in [1] to the description of the structure of concrete Banach algebras associated with automorphisms, namely the algebras generated by 'weighted composition operators' acting in Lebesgue spaces.

We establish a number of isomorphism results for the algebras investigated and in addition find out that the arising algebras are in a way qualitatively different. In particular when considering the operators in $L^\infty_\mu(\Omega, E)$ and in $L^1_\mu(\Omega, E)$ we *can* calculate their norms (see Theorems 8 and 10) while for the operators in $L^p_\mu(\Omega, E)$, $1 < p < \infty$ we have nothing like this. Moreover to obtain the isomorphism theorems for the algebras $B(A, T_g)$ in $L^\infty_\mu(\Omega, E)$, $L^1_\mu(\Omega, E)$ we do not need any information on the structure of the group of operators generating automorphisms while this structure (namely the amenability of the group G) is vital when we are investigating the operators in $L^p_\mu(\Omega, E)$, $1 < p < \infty$ (see Theorem 7 and Remark 6.1).

To make the presentation selfcontained we have to recall a number of notions and results from [1].

1.1. Property (*). *It was shown in* [1] *that one of the most important properties of the algebra* $B(A, T_g)$ *in the presence of which one can obtain the deep and fruitful theory of the subject is the next* property (*):

for any finite sum $b = \sum a_g T_g$, $a_g \in A$ *the following inequality holds*

$$\|b\| = \|\sum a_g T_g\| \geq \|a_e\|, \tag{1.3}$$

where e *is the identity of the group* G.

If an algebra $B(A, T_g)$ possesses the property (*) then for every $g_0 \in G$ there is correctly defined the mapping

$$N_{g_0} : \sum a_g T_g \to a_{g_0} \tag{1.4}$$

which can be extended up to the mapping

$$N_{g_0} : B(A, T_g) \to A. \tag{1.5}$$

1.2. Property (**). *One more important property is the following.*
We shall say that an algebra $B(A,T_g)$ which possesses the property (*) also possesses the property (**) *if*

$$B(A,T_g) \ni b = 0 \text{ iff } N_g(b) = 0 \text{ for every } g \in G \tag{1.6}$$

where N_g is the mapping introduced above.

In fact the presence of the properties (*) and (**) makes it possible to 'reestablish' an element $b \in B(A,T_g)$ via its 'Fourier' coefficients $N_g(b)$, $g \in G$ and we shall find out further that in many reasonable situations this 'reestablishing' can be carried out successfully.

If A is a C^*–algebra of operators containing the identity and acting in a Hilbert space H and $\{T_g\}_{g\in G}$ is a unitary representation of a group G in H then the C^*–algebra generated by A and $\{T_g\}_{g\in G}$ will be denoted by $C^*(A,T_g)$.
In the C^*–algebra situation we have (see [4], Theorems 12.8 and 12.4):

if G is a discrete amenable group and $C^*(A,T_g)$ possesses the property (*) then $C^*(A,T_g)$ possesses the property (**) as well.

The main reason why in the C^*–algebra case the property (**) (1.6) follows from the property (*) is that

the presence of the property (*) implies ([4], Theorem 12.8)

$$C^*(A,T_g) \cong A \times_{\hat{T}} G$$

where by $A \times_{\hat{T}} G$ we denote the crossed product of the algebra A by the group $\{\hat{T}_g\}_{g\in G}$ of its automorphisms (here G is considered as a discrete group).
Since in a Banach space case we do not have anything like the isomorphism mentioned above we have to check the property (**) even when $B(A,T_g)$ possesses the property (*). In a general situation (that is for an *arbitrary* discrete group of isometries $\{\hat{T}_g\}_{g\in G}$ with $T_gAT_g^{-1} = A$) the verification of the property (**) may be very sophisticated. The next Theorem 1 (proved in [1]) shows that in the case of a *locally compact commutative* group G and under a special assumption (which as it will be seen later is in fact rather common) the algebra $B(A,T_g)$ possesses the properties (*) and (**) simultaneously.

Theorem 1. *Let G be a locally compact commutative group. If for any finite set $F \subset G$ and any character $\chi \in \hat{G}$ there is satisfied the equality*

$$\|\sum_{g\in F} a_gT_g\| = \|\sum_{g\in F} a_g\chi(g)T_g\| \tag{1.7}$$

then the algebra $B(A,T_g)$ possesses the properties (*) and (**).

In [1] we have also established a close relation between the property (*) and the so-called *topological freedom* of the action of the group of automorphisms $\{\hat{T}_g\}$. So let as recall the latter notion.

1.3. Topologically free action. *Observe first that if $\{T_g\}_{g\in G}$ is a group of isometries satisfying (1.1) then evidently*

$$T_g\mathbb{Z}(A)T_g^{-1} = \mathbb{Z}(A) \tag{1.8}$$

where $\mathbb{Z}(A)$ is the center of A.
Let A be a certain Banach algebra isomorphic to $C(X,B)$ where X is a completely regular space

and B is a Banach algebra then

$$\mathbb{Z}(A) = C(X, \mathbb{Z}(B)). \tag{1.9}$$

Henceforth in this subsection we confine ourselves to the case

$$\mathbb{Z}(B) = \{cI\} \tag{1.10}$$

The reason justifying this choice was discussed in [1]. *Obviously if $B = L(E)$ is the Banach algebra of all linear bounded operators acting in a Banach space E then (1.10) is satisfied.*
So let $A \subset L(D)$ be a Banach algebra of operators isomorphic to $C(X, L(E))$ where X is a certain completely regular space and E and D are Banach spaces (thus $\mathbb{Z}(A) \cong C(X)$). Let $\{T_g\}_{g\in G}$ be a group of isometries satisfying (1.1). According to (1.8) the automorphisms $\hat{T}_g$ (1.2) preserve the center and henceforth we assume that their action on the center is given by

$$[\hat{T}_g(z)](x) = z(t_g^{-1}(x)), \quad z \in \mathbb{Z}(A), \ x \in X. \tag{1.11}$$

where $t_g : X \to X$ are some homeomorphisms of X.

Denote by X_g, $g \in G$ the set

$$X_g = \{x \in X : t_g(x) = x\}. \tag{1.12}$$

We say that the group G acts topologically freely *on A by automorphisms $\hat{T}_g$ (or on X by homeomorphisms t_g mentioned in (1.11)) if for any $g \in G$, $g \neq e$ the set X_g has an empty interior.*

One can observe that G acts topologically freely iff for any finite set $\{g_1, ..., g_n\} \subset G$ $(g_i \neq e)$ the set $[\cup_{i=1}^n X_{g_i}]$ has an empty interior.

Just as in [4], *12.13 and 12.13' it can be shown that the foregoing definition is equivalent to the next one: the action of G is said to be topologically free if for any finite set $\{g_1, ..., g_k\} \subset G$ and a non empty open set $U \subset X$ there exists a point $x \in U$ such that all the points $t_{g_i}(x)$, $i = 1, ..., k$ are distinct.*

Since X is Hausdorff the latter definition is also equivalent to the following: the action of G is said to be topologically free if for any finite set $\{g_1, ..., g_k\} \subset G$ and a non empty open set $U \subset X$ there exists a non empty open set $V \subset U$ such that

$$t_{g_i}(V) \cap t_{g_j}(V) = \emptyset \quad i, j \in \overline{1, k}, \ i \neq j. \tag{1.13}$$

In [1] we proved the following

Theorem 2. *If G acts topologically freely then $B(A, T_g)$ possesses the property (*).*

1.4. Regular representation of an algebra and a group of automorphisms. *Let us also recall one more algebra (examined in* [1]*) where the properties (*) and (**) can be checked easily* — the regular representation of an algebra A and a group of automorphisms $\{\hat{T}_g\}_{g\in G}$.
Namely let $A \subset L(D)$ be a certain Banach algebra and $\{\hat{T}_g\}_{g\in G}$ be a certain group of its automorphisms (G is an arbitrary *group that is* not necessarily commutative*).*
Denote by H any of the spaces $l^p(G, D), 1 \le p \le \infty$ or $l_0(G, D)$ (here $l_0(G, D)$ is the space of vector valued functions on G having values in D and tending to zero at infinity (with the sup-norm)).
Set the operators $V_{g_0} : H \to H$ by the formula

$$(V_{g_0}\xi)(g) = \xi(gg_0), \quad g, g_0 \in G \tag{1.14}$$

and consider the algebra $\bar{A} \subset L(H)$ isomorphic (as a Banach algebra) to A and given by

$$(\bar{a}\xi)(g) = \hat{T}_g(a)\xi(g), \qquad a \in A. \tag{1.15}$$

Routine computation shows that with this notation we have

$$V_g \bar{a} V_g^{-1} = \overline{\hat{T}_g(a)}$$

which in view of the isomorphism between A and $\bar{A}$ means that the operators $V_g, g \in G$ given by (1.14) generate the automorphisms $\hat{T}_g$ of $\bar{A}$.

The algebra $B(\bar{A}, V_g) \subset L(H)$ is called the (right) regular representation *corresponding to the algebra A and the group of automorphisms $\{\hat{T}_g\}_{g\in G}$ (in fact we have the series of representations depending on the type of the space H chosen).*

In [1] we have proved that

the algebra $B(\bar{A}, V_g)$ possesses the properties (*), (**) and (1.7) (for every H considered).

2. Metrically free action and topologically free action

2.1. *Let (Ω, μ) be a space with a σ−additive σ−finite measure μ, H be a certain Banach space and $L^p_\mu(\Omega, H)$, $1 \le p \le \infty$ be the spaces of (equivalence classes) of measurable functions $f : \Omega \to H$ bounded with respect to the norms*

$$\|f\| = \left[\int_\Omega |f(x)|^p \, d\mu(x)\right]^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\| = \operatorname{esssup}_\Omega |f|, \qquad p = \infty$$

where $|\cdot|$ is the norm in H (for details see, for example, Dunford, Schwarts [5]).
Consider an algebra $A \subset L(D)$ isomorphic to $L^\infty_\mu(\Omega, L(E))$ where D and E are Banach spaces. If $\{T_g\}_{g\in G}$ is a group of isometrics of D satisfying (1.1) then the automorphisms $\hat{T}_g$ (given by (1.2)) generate the mappings $\alpha_g : \Sigma \to \Sigma$ (Σ is the set of (equivalence classes) of measurable subsets of Ω) defined in the following way.

Observe that for the algebra considered the center $\mathbb{Z}(A) \cong L^\infty_\mu(\Omega)$. Let χ_Δ be the element of $\mathbb{Z}(A)$ corresponding to the characteristic function $\chi_\Delta(\omega)$ of a certain set $\Delta \in \Sigma$.
Since ${\chi_\Delta}^2 = \chi_\Delta$ it follows that $\hat{T}_g(\chi_\Delta)$ is a projection belonging to $\mathbb{Z}(A)$ (non zero iff $\chi_\Delta \neq 0$) and we have

$$\hat{T}_g(\chi_\Delta) = \chi_{\tilde{\Delta}} \tag{2.1}$$

for some $\tilde{\Delta} \in \Sigma$.
We set

$$\alpha_g(\Delta) = \tilde{\Delta}. \tag{2.2}$$

The substitution for the topologically free action of G (see 1.3) in the situation under consideration is the so-called metrically free action. Here it is.

2.2. *We say that the group G acts* metrically freely *on A (considered in 2.1) by automorphisms $\hat{T}_g$ (or on Σ by the mappings α_g) if for any finite set $\{g_1, ..., g_k\} \subset G$ and any $\Delta \in \Sigma$ with $\mu(\Delta) > 0$ there exists a set $\Delta' \in \Sigma$ such that*
(i) $\mu(\Delta') > 0$,
(ii) $\Delta' \subset \Delta$,

(iii) $\mu(\alpha_{g_i}(\Delta')\cap\alpha_{g_j}(\Delta'))=0, \quad i,j\in\overline{1,k},\ i\neq j.$

It is worth mentioning that from a certain point of view the notion of the metrically free action of G just introduced 'coincides' with the notion of the topologically free action.

Indeed.
The algebra $L^\infty_\mu(\Omega)$ is a commutative C^*–algebra (with the natural involution). Let M be its maximal ideal space then

$$L^\infty_\mu(\Omega)\cong C(M)$$

where the isomorphism is established by means of the Gelfand transform. The mentioned 'coincidence' between the topologically and metrically free actions is established in the next

Theorem 3. *The metrically free action of the automorphisms $\hat{T}_g$ on $L^\infty_\mu(\Omega)$ corresponds to the topologically free action of the automorphisms $\check{T}_g$ on $C(M)$ (induced by the automorphisms $\hat{T}_g$ and the isomorphism $L^\infty_\mu(\Omega)\cong C(M)$).*

Now the analogue to Theorem 2 for the measurable case considered is

Theorem 4. *Let A and T_g, $g\in G$ be those considered in 2.1. If G acts metrically freely then $B(A,T_g)$ possesses the property (*).*

3. Example 1. Operators in $L^p(\Omega,E), 1<p<\infty$

3.1. *Let (Ω,μ) be a space with a $\sigma-$additive $\sigma-$finite measure μ. Consider the space $D=L^p_\mu(\Omega,E)$, $1<p<\infty$. Let $A=L^\infty_\mu(\Omega,L(E))\subset L(D)$ be the algebra of* multiplication operators *defined by*

$$(af)(x)=a(x)f(x),\quad a\in A,\ f\in D \tag{3.1}$$

and let $\alpha_g:\Omega\to\Omega$, $g\in G$ be a group of measurable mappings preserving the equivalence class of μ. By T_g we denote the isometry of D defined by

$$(T_gf)(x)=\left[\frac{d\alpha_g^{-1}(\mu)}{d\mu}\right]^{\frac{1}{p}}f(\alpha_g^{-1}(x)) \tag{3.2}$$

where $\frac{d\alpha_g^{-1}(\mu)}{d\mu}$ is the Radon-Nikodim derivative of $\alpha_g^{-1}(\mu)$ with respect to μ.
One can easily verify that T_g satisfies (1.1) and the mentioned mappings α_g coincide with those described in 2.1.

Let $B(A,T_g)\subset L(D)$ be the algebra generated by A and $\{T_g\}_{g\in G}$.

Theorem 5. *Let $B(A,T_g)$ be the algebra described in 3.1 and $B(\bar{A},V_g)$ be the corresponding regular representation in the space $H=l^p(G,L^p_\mu(\Omega,E))$. If G acts metrically freely then the mapping*

$$B(A,T_g)\to B(\bar{A},V_g)$$

defined by

$$a\to\bar{a},\quad a\in A$$

$$T_g\to V_g,\quad g\in G$$

is norm decreasing.

Theorem 6. *Let $B(A, T_g)$ be the algebra described in 3.1 and $B(\bar{A}, V_g)$ be the corresponding regular representation in the space $H = l^p(G, L^p_\mu(\Omega, E))$. If G is amenable then the mapping*

$$B(\bar{A}, V_g) \to B(A, T_g)$$

generated by the mappings

$$\bar{a} \to a, \quad a \in A \tag{3.3}$$

$$V_g \to T_g, \quad g \in G \tag{3.4}$$

is norm decreasing.

We can summarize the results obtained in

Theorem 7. *Let $B(A, T_g)$ and $B(\bar{A}, V_g)$ be those considered in the lemma.*
If G is amenable and acts metrically freely then

$$B(A, T_g) \cong B(\bar{A}, V_g)$$

where the isomorphism is given by (3.3) and (3.4).
In particular the algebra $B(A, T_g)$ possesses the properties (*) and (**) and (1.7).

4. Example 2. Operators in $L^\infty_\mu(\Omega, E)$

4.1. *Let $D = L^\infty_\mu(\Omega, E)$ (where Ω is a space with a $\sigma-$additive $\sigma-$finite measure μ) and $A = L^\infty_\mu(\Omega, L(E))$ be the algebra of multiplication operators (see (3.1)) and $\alpha_g : \Omega \to \Omega,\ g \in G$ be a group of measurable mappings preserving the equivalence class of μ. By T_g we denote an isometry of D given by*

$$(T_g f)(x) = f(\alpha_g^{-1}(x)). \tag{4.1}$$

Let $B(A, T_g)$ be the algebra generated by A and $\{T_g\}_{g \in G}$.

For any fixed finite set $F \subset G$ we denote by $B_F(D)$ and $S_F(D)$ respectively the sets

$$B_F(D) = \{\{f_g\}_{g \in F} : f_g \in D,\ \|f_g\| \leq 1,\ g \in F\}, \tag{4.2}$$

$$S_F(D) = \{\{f_g\}_{g \in F} : f_g \in D,\ \|f_g\| = 1,\ g \in F\}. \tag{4.3}$$

Theorem 8. *Let $B(A, T_g)$ be the algebra introduced above. If G acts metrically freely then for any finite F we have*

$$\|\sum_{g \in F} a_g T_g\| = \sup_{\{f_g\}_{g \in F} \in S_F(D)} \|\sum_{g \in F} a_g f_g\| = \sup_{\{f_g\}_{g \in F} \in B_F(D)} \|\sum_{g \in F} a_g f_g\| \tag{4.4}$$

where $B_F(D)$ and $S_F(D)$ are defined by (4.2) and (4.3).

Remark 1.

(1) If $E = \mathbb{C}$ (that is $D = L^\infty_\mu(\Omega)$ and $A = L^\infty_\mu(\Omega)$) then (4.4) implies

if G acts metrically freely then

$$\|\sum_F a_g T_g\| = \operatorname{esssup}_\Omega \sum_F |a_g(x)|$$

Indeed on the one hand

$$\sup_{\{f_g\} \in B_F(D)} \|\sum_{g \in F} a_g f_g\| \leq \operatorname{esssup}_\Omega \sum_F |a_g(x)|$$

and to obtain the opposite inequality just set

$$f_g(x) = \begin{cases} [\arg a_g(x)]^{-1} & , \quad \text{if} \quad a_g(x) \neq 0 \\ 1 & , \quad \text{if} \quad a_g(x) = 0 \end{cases}$$

(since $a_g \in L^\infty_\mu(\Omega)$ we have that $f_g \in L^\infty_\mu(\Omega)$ as well).

(2) The equality (4.4) also shows that

if G acts metrically freely then

$$\|\sum_F a_g T_g\| = \|\tilde{b}_F\|$$

where

$$\tilde{b}_F : D_1 \times D_2 \times \ldots \times D_{|F|} \to D, \qquad D_i = D$$

is given by

$$\tilde{b}_F(\xi_1, \ldots, \xi_{|F|}) = a_{g_1}\xi_1 + \ldots + a g_{|F|}\xi_{|F|} \tag{4.5}$$

$(\{g_1, \ldots, g_{|F|}\} = F)$.

(3) The preceding remark leads in turn to the next observation.

Since $a_g \in L^\infty_\mu(\Omega, L(E))$, $g \in F$ it follows (by the structure of $\tilde{b}_F$) that

$$\tilde{b}_F \in L^\infty_\mu(\Omega, L(\tilde{E}, E))$$

where $\tilde{E} = E_1 \times E_2 \times \ldots \times E_{|F|}$, $E_i = E$.
But this means that

$$\|\tilde{b}_F(\cdot)\| \in L^\infty_\mu(\Omega)$$

and

$$\|\tilde{b}_F\| = \operatorname{esssup}_\Omega \|\tilde{b}_F(x)\|$$

which along with the preceding remark (2) implies

if G acts metrically freely then

$$\|\sum_F a_g T_g\| = \operatorname{esssup}_\Omega \sup_{\{f_g\} \in B_F(E)} \|\sum_{g \in F} a_g(x) f_g\| =$$
$$\operatorname{esssup}_\Omega \sup_{\{f_g\} \in S_F(E)} \|\sum_{g \in F} a_g(x) f_g\| \tag{4.6}$$

thus strengthening the statement of Theorem 8.

Theorem 9. *Let $B(A, T_g)$ be the algebra described in 4.1 and $B(\bar{A}, V_g)$ be the corresponding regular representation in the space*

$$H = l_0(G, L^\infty_\mu(\Omega, E)) \quad (\text{or} \quad l^\infty(G, L^\infty_\mu(\Omega, E))).$$

If G acts metrically freely then

$$B(A, T_g) \cong B(\bar{A}, V_g)$$

where the isomorphism is generated by the mappings

$$a \to \bar{a}, \quad a \in A$$

$$T_g \to V_g, \quad g \in G$$

and in particular $B(A, T_g)$ possesses the properties (*) and (**) and (1.7).

5. Example 3. Operators in $L^1_\mu(\Omega, E)$

Our final example deals with the space L^1. By means of duality this case can be reduced to the L^∞ situation already studied.

5.1. *Let (Ω, μ) be the space considered in 3.1, $D = L^1_\mu(\Omega, E)$ and $A = L^\infty_\mu(\Omega, L(E))$ be the algebra of operators defined by (3.1) and T_g be defined by (3.2) (with $p = 1$).*
Let $B(A, T_g) \subset L(D)$ be the algebra generated by A and $\{T_g\}_{g\in G}$.

Theorem 10. *Let $B(A, T_g)$ be the algebra introduced in 5.1. If G acts metrically freely then*

$$\|\sum_{g\in F} a_g T_g\| = \operatorname{esssup}_\Omega \sup_{\{f_g\}\in B_F(E^*)} \|\sum_{g\in F}[a_g(\alpha_g(x))]^* f_g\| =$$

$$\operatorname{esssup}_\Omega \sup_{\{f_g\}\in S_F(E^*)} \|\sum_{g\in F}[a_g(\alpha(x))]^* f_g\| \tag{5.1}$$

Remark 2. *(cf. Remark 1 (1)).*
If $E = \mathbb{C}$ and G acts metrically freely then

$$\|\sum_F a_g T_g\| = \operatorname{esssup}_\Omega \sum_F |a_g(\alpha_g(x))|.$$

And finally the analogue to Theorem 9 here is

Theorem 11. *Let $B(A, T_g)$ be the algebra described in 5.1 and $B(\bar{A}, V_g)$ be the corresponding regular representation in the space $l^1(G, L^1_\mu(\Omega, E))$. If G acts metrically freely then*

$$B(A, T_g) \cong B(\bar{A}, V_g)$$

where the isomorphism is generated by the mappings

$$a \to \bar{a}, \quad a \in A$$

$$T_g \to V_g, \quad g \in G$$

and in particular $B(A, T_g)$ possesses the properties (*) and (**) and (1.7).

6. Final remarks. Isomorphism theorems.

Now we would like to observe certain interrelations between the examples considered and the Isomorphism Theorem ([4], Corollary 12.17).

6.1. *(1) Observe that in Examples 2, 3 we did not use any information about the group G thus the group in these examples is* not necessarily amenable.

(2) The essentially different picture is drawn in Example 1.
Here
(i) if G acts metrically freely then
$B(\bar{A}, V_g)$ is a representation of $B(A, T_g)$ for any *G (*not necessarily amenable*)*
(Theorem 5).

While
(ii) if G is amenable then
$B(A, T_g)$ is a representation of $B(\bar{A}, V_g)$ for an arbitrary *action of G (*not necessarily metrically free*).*
(Theorem 6).

Thus in these examples the metrical freedom of the action of G and the amenability of G are lying in a sense opposite each other.
If G acts metrically freely then $B(A,T_g)$ is 'larger' than $B(\bar{A},V_g)$ (see (i)).
And
if G is amenable then $B(\bar{A},V_g)$ is 'larger' than $B(A,T_g)$ (see (ii)).

Both these algebras 'coincide' if G acts metrically freely and is amenable (Theorem 7).

(3) Consideration of Example 1 leads to certain Isomorphism Theorems *which (just as it was done in [4], Corollary 12.17) establish the isomorphism between* essentially spatially different *operator algebras (thus wiping off the spaces where these operators act).*
For example.
Let (Ω,μ_i), $i=1,2$ be two spaces with $\sigma-$additive $\sigma-$finite separable measures μ_1 and μ_2 absolutely continuous with respect to each other (thus $L^\infty_{\mu_1}(\Omega,L(E)) \cong L^\infty_{\mu_2}(\Omega,L(E))$) and let $\{\alpha_g\}_{g\in G}$ be a group of measurable mappings of Ω preserving the equivalence classes of μ_1 and μ_2. Consider the spaces $D_i = L^p_{\mu_i}(\Omega,E)$, $ß = 1,2$. Let $A_i = L^\infty_{\mu_1}(\Omega,L(E)) \subset L(D_i)$ be the algebras of multiplication operators defined by (3.1) and T^i_g, $i=1,2$ be the isometries of D_i defined by (3.2) (with $\mu=\mu_i$) and $B(A,T^i_g)$ be the algebras generated by A_i and $\{T^i_g\}_{g\in G}$.
The Isomorphism Theorem related to Example 1 is stated as follows:

If E is a separable Banach space and G is a countable amenable group acting metrically freely then $B(A_1,T^1_g)$ and $B(A_2,T^2_g)$ are isomorphic (as Banach algebras) and the isomorphism is established by the natural isomorphism

$$A_1 \cong A_2$$

and the mapping

$$T^1_g \to T^2_g.$$

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