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A PRIORI ESTIMATES AND BLOW-UP OF SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL INEQUALITIES

Using the test-function method developed by Mitidieri and Pohozaev we obtain global a priori estimates and prove blow-up results for semilinear higher–order evolution inequalities in exterior domain.

1. Introduction

This paper is devoted to nonexistence of global solutions to some evolution equations and inequalities. The number of publications on nonexistence for evolution problems is so great that we cannot cite all of them. For parabolic problems let us only mention the classical book by Samarskii, Galaktionov, Kurdyumov & Mikhailov [16], the survey by Levine [10] and the more recent one by Deng & Levine [2]. The main results for hyperbolic problems can be found in the survey by John [4] and the papers by Del Santo, Georgiev & Mitidieri [1] and by Veron & Pohozaev [17].

Finally we refer the interested readers to the celebrated monograph by Mitidieri and Pohozaev [12], where the test-function method is systematically applied to quasilinear elliptic (see also [5, 6, 3, 7]), parabolic (see also [9, 18]) and hyperbolic (see also [17, 8]) partial differential inequalities and systems of such inequalities.

2. Main result

Let $R > 0$, $k \in \mathbb{N}$. Let us introduce the domain $\Omega = \mathbb{R}^N \setminus B_R$ (where $N \geq 3$ and $B_R = \{|x| \leq R\}$) and consider the problem

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \geq |u|^q, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) \geq 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0, & x \in \Omega. \end{cases} \tag{1}$$

Definition 1. Let $u(x,t) \in C(\overline{\Omega} \times [0,\infty))$ and the locally integrable traces $\frac{\partial^i u}{\partial t^i}(x,0)$, $i = 1,\dots,k-1$, are well defined. The function $u(x,t)$ is called a weak solution to problem (1) if, for any nonnegative test-function $\varphi(x,t) \in W^{2,k}_\infty(\Omega \times (0,\infty))$ with compact support, such that $\varphi|_{\partial\Omega \times (0,\infty)} = 0$, the integral inequality

$$\int_0^\infty \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n}\, dxdt + \int_0^\infty \int_\Omega u \left((-1)^k \frac{\partial^k \varphi}{\partial t^k} - \Delta\varphi \right) dxdt \geq \int_0^\infty \int_\Omega |u|^q \varphi\, dxdt$$
$$+ \sum_{i=1}^{k-1} (-1)^i \int_\Omega \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x,0) \frac{\partial^i \varphi}{\partial t^i}(x,0)\, dx + \int_\Omega \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \varphi(x,0)\, dx \tag{2}$$

holds.

Theorem 1. *For*

$$1 < q \leq q_k^* = \frac{N + 2/k}{N - 2 + 2/k}$$

problem (1) *has no nontrivial global solution.*

This theorem includes, among others, the sharp results for parabolic ($k = 1$, Fujita–Hayakawa's critical exponent $q_1^* = 1 + \frac{2}{N}$, see also [14]) and hyperbolic ($k = 2$, Kato's critical exponent $q_2^* = \frac{N+1}{N-1}$, see also [17]) problems. It is interesting, that formally passing to the limit as $k \to \infty$ in Theorem 1 we arrive at the sharp elliptic critical exponent $q_\infty^* = \frac{N}{N-2}$ (see, for example, [5, 11, 12, 7] and the references therein).

3. Auxiliary estimates

In this section we obtain some estimates depending on the parameter ρ, $\rho \to \infty$. These estimates play fundamental role in the test function method.

Let us consider the "standard cut-off function" $\zeta(y) \in C^\infty(\mathbb{R}_+)$ with the following properties:

$$0 \le \zeta(y) \le 1, \qquad \zeta(y) = \begin{cases} 1, & \text{if } 0 \le y \le 1, \\ 0, & \text{if } y \ge 2. \end{cases}$$

For the function $\eta(y) = (\zeta(y))^{kp_0}$ with some positive p_0 and $k \in \mathbb{N}$ by direct calculation one can obtain the estimates (for $1 < p \le p_0$)

$$|\eta'(y)|^p = (kp_0)^p \zeta^{kp_0(p-1)} \zeta^{kp_0-p} |\zeta'|^p \le c_\eta \eta^{p-1}(y),$$
$$|\eta''(y)|^p \le (kp_0)^p \zeta^{kp_0(p-1)} \zeta^{kp_0-2p}((kp_0-1)|\zeta'|^2 + \zeta|\zeta''|)^p \le c_\eta \eta^{p-1}(y), \quad \ldots$$
$$|\eta^{(k)}(y)|^p \le c_\eta \eta^{p-1}(y)$$

with a positive constant c_η.

Now let us introduce the change of variables $y = t/\rho^\theta$, with $\theta > 0$, $\rho > 2R$. For the function $\eta(t/\rho^\theta)$ we have

$$\operatorname{supp}\left|\eta\left(\frac{t}{\rho^\theta}\right)\right| = \{t < 2\rho^\theta\}, \qquad \operatorname{supp}\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right| = \{\rho^\theta < t < 2\rho^\theta\},$$

and

$$\int\limits_{\operatorname{supp}\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|} \frac{\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|^p}{\eta^{p-1}(t/\rho^\theta)}\, dt \le c_\eta \rho^{-\theta(kp-1)}. \tag{3}$$

The parameter θ will be chosen later.

For the variable x, $|x| = r \ge R$, we introduce the functions $\eta(r/\rho)$,

$$\xi(x) \equiv \xi(r) = \frac{1}{R^s} - \frac{1}{r^s}, \tag{4}$$

and

$$\psi_\rho(x) \equiv \psi_\rho(r) = \left(\frac{1}{R^s} - \frac{1}{r^s}\right)\eta\left(\frac{r}{\rho}\right). \tag{5}$$

Here we will put $s = N - 2$. It is evidently, that $\psi_\rho = 0$ and $\frac{\partial\psi_\rho}{\partial r} \ge 0$ as $r = R$.

For the derivatives of the function $\psi_\rho(r)$ (as $r > 2R$) we have:

$$\left|\frac{\partial\psi_\rho}{\partial r}\right|^p \le \left|\frac{s}{r^{s+1}}\eta\left(\frac{r}{\rho}\right) + \left(\frac{1}{R^s} - \frac{1}{r^s}\right)\eta'\left(\frac{r}{\rho}\right)\frac{1}{\rho}\right|^p \le c\eta^{p-1}\left(\frac{r}{\rho}\right)\frac{1}{R^{ps}r^p}\left(1 + \frac{r^p}{\rho^p}\right),$$

$$\left|\frac{\partial^2\psi_\rho}{\partial r^2}\right|^p \le \left|-\frac{s(s+1)}{r^{s+2}}\eta\left(\frac{r}{\rho}\right) + \frac{2s}{r^{s+1}\rho}\eta'\left(\frac{r}{\rho}\right) + \left(\frac{1}{R^s} - \frac{1}{r^s}\right)\frac{1}{\rho^2}\eta''\left(\frac{r}{\rho}\right)\right|^p$$
$$\le c\eta^{p-1}\left(\frac{r}{\rho}\right)\frac{1}{R^{sp}r^{2p}}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right),$$

here c does not depend on r and ρ. Using these estimates we arrive at the inequality for the Laplace operator:

$$|\Delta\psi_\rho(x)|^p = \left|\frac{\partial^2\psi_\rho}{\partial r^2} + \frac{N-1}{r}\frac{\partial\psi_\rho}{\partial r}\right|^p \le c\left|\frac{\partial^2\psi_\rho}{\partial r^2}\right|^p + c\frac{1}{r^p}\left|\frac{\partial\psi_\rho}{\partial r}\right|^p$$
$$\le c\eta^{p-1}\left(\frac{r}{\rho}\right)\frac{1}{R^{sp}r^{2p}}\left(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\right) \le c\psi_\rho^{p-1}(x)\frac{1}{r^{2p}}\left(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\right). \quad (6)$$

Now we take $s = N-2$. Due to $\Delta\left(\frac{1}{r^{N-2}}\right) = 0$ for $r \ne 0$ we have $\Delta\psi_\rho = 0$ for $r < \rho$ and $\operatorname{supp}|\Delta\psi_\rho| = \{\rho < r < 2\rho\}$. On the set $\operatorname{supp}|\Delta\psi_\rho|$ the estimate $1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}} \le c$ holds, where c does not depend on r and ρ. Therefore, it follows from (6) (for $\rho < r < 2\rho$) that

$$|\Delta\psi_\rho(x)|^p \le c\psi_\rho^{p-1}(x)\frac{1}{\rho^{2p}};$$

and we get

$$\int_{\operatorname{supp}|\Delta\psi_\rho|} \frac{|\Delta\psi_\rho(x)|^p}{\psi_\rho^{p-1}(x)}\,dx \le c\int_\rho^{2\rho}\frac{\psi_\rho^{p-1}(x)}{\psi_\rho^{p-1}(x)}\frac{r^{N-1}}{\rho^{2p}}\,dr \le c_\psi\rho^{-2p+N}. \quad (7)$$

Finally, for the general test-function

$$\varphi_\rho(x,t) = \eta\left(\frac{t}{\rho^\theta}\right)\psi_\rho(x) \quad (8)$$

we obtain the inequality

$$\int_{\operatorname{supp}|\Delta\varphi_\rho|}\frac{|\Delta\varphi_\rho(x,t)|^p}{\varphi_\rho^{p-1}(x,t)}\,dx\,dt \le \int_0^{2\rho^\theta}\eta(t/\rho^\theta)\,dt\int_{\operatorname{supp}|\Delta\psi_\rho|}\frac{|\Delta\psi_\rho|^p}{\psi_\rho^{p-1}}\,dx \le c_\varphi\rho^{\theta-2p+N}. \quad (9)$$

Analogously, using (3), we obtain:

$$\int_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|}\left|\frac{\partial^k\varphi_\rho(x,t)}{\partial t^k}\right|^p\frac{1}{\varphi_\rho^{p-1}(x,t)}\,dx\,dt$$
$$\le \int_{\operatorname{supp}|\frac{d^k\eta(t/\rho^\theta)}{dt^k}|}\left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|^p\frac{1}{\eta^{p-1}(t/\rho^\theta)}\,dt\int_{R<|x|<2\rho}\psi_\rho(x)\,dx$$
$$\le c_\eta\rho^{-\theta(kp-1)}c\int_R^{2\rho}r^{N-1}\,dr \le c_\varphi\rho^{N-\theta(kp-1)}. \quad (10)$$

For $\theta = 2/k$ the powers in these estimates are equal:

$$N - \theta(kp-1) = \theta - 2p + N \equiv -2p + N + 2/k.$$

Finally, we have

$$\int_{\operatorname{supp}|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k} - \Delta\varphi_\rho|}\frac{\left|(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k} - \Delta\varphi_\rho\right|^p}{\varphi_\rho^{p-1}}\,dxdt \le c_0\rho^{-2p+N+2/k}. \quad (11)$$

4. Proof of Theorem 1

Let $u(x,t)$ be a global nontrivial solution of problem (1). From Definition 1 with the test function $\varphi(x,t)=\varphi_\rho(x,t)$, defined by (8) with $p=q'>1$ and $\theta=2/k$, using the equalities $\frac{\partial^i\varphi_\rho}{\partial t^i}(x,0)\equiv 0$, $i=1,\dots,k-1$, we obtain

$$\int\limits_\Omega \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx+\int\limits_0^\infty\int\limits_\Omega |u|^q\varphi_\rho\,dxdt\le-\int\limits_0^\infty\int\limits_{\partial B_R} u\frac{\partial\varphi_\rho}{\partial r}\,dxdt+\int\limits_{\operatorname{supp}|A\varphi_\rho|} uA\varphi_\rho\,dxdt, \quad (12)$$

where

$$A\varphi_\rho=(-1)^k\frac{\partial^k\varphi_\rho}{\partial t^k}-\Delta\varphi_\rho.$$

As it was mentioned above $\frac{\partial\psi_\rho}{\partial r}|_{r=R}\ge 0$, so that $\frac{\partial\varphi_\rho}{\partial r}|_{r=R}\ge 0$ and the first integral in the right hand side is nonpositive due to our assumption $u|_{\partial\Omega\times(0,\infty)}\ge 0$.

As for the last integral in (12), using the Hölder inequality, we find that

$$\int\limits_\Omega \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx+\int\limits_0^\infty\int\limits_\Omega |u|^q\varphi_\rho\,dxdt$$

$$=\int\limits_\Omega \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx+\int\limits_{\operatorname{supp}|A\varphi_\rho|}|u|^q\varphi_\rho\,dxdt+\int\limits_{\varphi_\rho(x,t)=\xi(x)}|u|^q\xi\,dxdt$$

$$\le\int\limits_{\operatorname{supp}|A\varphi_\rho|}|u|\cdot|A\varphi_\rho|\,dxdt\le\left(\int\limits_{\operatorname{supp}|A\varphi_\rho|}|u|^q\varphi_\rho\,dxdt\right)^{1/q}\left(\int\limits_{\operatorname{supp}|A\varphi_\rho|}\frac{|A\varphi_\rho|^{q'}}{\varphi_\rho^{q'-1}}\,dxdt\right)^{1/q'}. \quad (13)$$

Now, using the estimate (11) (with $p=q'$) for the last integral in the right-hand side, we have

$$\int\limits_{\varphi_\rho(x,t)=\xi(x)}|u|^q\xi(x)\,dxdt\le\int\limits_{\operatorname{supp}|A\varphi_\rho|}\frac{|A\varphi_\rho|^{q'}}{\varphi^{q'-1}}\,dxdt\le c_0\rho^{-2q'+N+2/k}. \quad (14)$$

Now we pass to the limit as $\rho\to\infty$. In the case

$$-2q'+N+2/k\le 0 \quad (15)$$

this implies

$$\int\limits_0^\infty\int\limits_\Omega |u|^q\xi\,dxdt\le c_0.$$

Then by the inequality $\varphi_\rho\le\xi$ and taking into account the general properties of Lebesgue integral we have

$$\int\limits_{\operatorname{supp}|A\varphi_\rho|}|u|^q\varphi_\rho\,dxdt\le\int\limits_{\operatorname{supp}|A\varphi_\rho|}|u|^q\xi\,dxdt=\varepsilon(\rho)\to 0$$

as $\rho\to\infty$.

Then from the inequality (13) we finally get

$$\int\limits_{\varphi_\rho(x,t)=\xi(x)}|u|^q\xi\,dxdt\le\varepsilon^{1/q}(\rho)c_0^{1/q'}\to 0$$

as $\rho \to \infty$, and $\int_0^\infty \int_\Omega |u|^q \xi \, dxdt = 0$, that is, the solution $u(x,t)$ must be trivial under condition (15), which is equivalent to the condition of Theorem 1. □

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References

[1] Del Santo D., Georgiev V., Mitidieri E., *Global existence of the solutions and formation of singularities for a class of hyperbolic systems*, In: *"Geometric Optics and Related Topics"* (Eds. F. Colombini & N. Lerner), Progress in Nonlinear Differential Equations and Their Applications, Vol. 32, 117-140. Birkhäuser, Boston, 1997.

[2] Deng K., Levine H. A., *The role of critical exponent in blow-up theorems: the sequel*, J. Math. Anal. Appl. **243** (2000), 85–126.

[3] Galakhov E., Some nonexistence results for quasilinear elliptic problems, J. Math. Anal. Appl. 252 (2000) 256–277.

[4] John F., *Nonlinear wave equations, formation of singularities*, University Lecture Ser. **2** (1990), Amer. Math. Soc.

[5] Kurta V. V., *On the nonexistence of positive solutions to semilinear elliptic equations*, Proc. Steklov Inst. Math. **227** (1999), 162–169.

[6] Kartsatos A., Kurta V. V. *On nonexistence of entire solutions to quasilinear equations*, Dokl. Russ. Acad. Sci. **371** (2000), 591–593.

[7] Laptev G. G., *Nonexistence of global positive solutions for systems of semilinear elliptic inequalities in cone*, Russian Acad. Sci. Izv. Math. **64** (2000), 108–124.

[8] Laptev G. G., *On nonexistence for a class of singular semilinear differential inequalities*, Proc. Steklov Inst. Math. **232** (2001), 223–235.

[9] Laptev G. G., *Nonexistence of solutions to semilinear parabolic inequalities in cones*, Mat. Sb. (2001), to appear.

[10] Levine H. A., *The role of critical exponents in blow-up theorems*, SIAM Rev. **32** (1990), 262–288.

[11] Mitidieri E., Pohozaev S. I., *Nonexistence of positive solutions for quasilinear elliptic problems on* $\mathbb{R}^N$, Proc. Steklov Inst. Math. **227** (1999), 192–222.

[12] Mitidieri E., Pohozaev S. I., *A priori Estimates and Nonexistence of Solutions to Nonlinear Partial Differential Equations and Inequalities*, Proc. Steklov Inst. Math. **234** (2001).

[13] Mitidieri E., Pohozaev S. I., *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on* $\mathbb{R}^n$, J. Evolution Equations **1** (2001), 189-Ц220.

[14] Mochizuki K., Suzuki R., *Critical exponent and critical blow up for quasi-liner parabolic equations*, Israel J. Math. **98** (1997), 141–156.

[15] Pohozaev S. I., Tesei A., *Blow-up of nonnegative solutions to quasilinear parabolic inequalities*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), 99–109.

[16] Samarskii A. A., Galaktionov V. A., Kurdyumov S. P., Mikhailov A. P., *Blow-up in Quasilinear Parabolic Equations*, Nauka, Moscow, 1987 (in Russian). English translation: Walter de Gruyter, Berlin/New York, 1995.

[17] Veron L., Pohozaev S. I., *Blow-up results for nonlinear hyperbolic inequalities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4). **29** (2000), 393–420.

[18] Zhang Q., *Blow-up results for nonlinear parabolic equations on manifolds*, Duke Math. J. **97** (1999), 515–539.

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