Ученые записки Таврического национального университета им. В. И. Вернадского
Серия "Математика. Механика. Информатика и Кибернетика"N 1 (2002) 19 – 21
УДК 517.98

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COMMON INVARIANT SUBSPACES FOR COMMUTING CONTRACTIONS

Keywords: Contractions, Dual algebras, Functional Calculus

The results presented here are contained in [6].

Let $\mathfrak{H}$ be a (complex, separable, infinite dimensional) Hilbert space and $\mathfrak{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. Denote by $\mathbb{D}$ the (open) unit disk in the complex plane $\mathbb{C}$ and by $\mathbb{T}$ the unit circle. By the *polydisk* $\mathbb{D}^N \subset \mathbb{C}^N$ we mean the Cartesian product of N copies of $\mathbb{D}$ and by $H^\infty(\mathbb{D}^N)$ the algebra of bounded analytic functions on the polydisk $\mathbb{D}^N$. A set $\Delta \subset \mathbb{D}^N$ is said to be *dominating for* $\mathbb{T}^N$ if

$$\sup_{z\in\Delta} |h(z)| = \|h\|_\infty, \quad \text{for all } h \in H^\infty(\mathbb{D}^N).$$

Several authors (cf., [2], and [9]) have constructed *representations* of $H^\infty(\mathbb{D}^N)$. That is an algebra homomorphism Φ from $H^\infty(\mathbb{D}^N)$ into $\mathfrak{L}(\mathfrak{H})$. A representation Φ is said to be *generated by* an N–tuple of commuting contractions $T = (T_1 \dots, T_N)$ if $\Phi_T(p) := \Phi(p) = p(T)$, for any polynomial p in N complex variables. We shall say a representation is *contractive* if $\|\Phi_T(h)\| \le \|h\|_\infty$, for all $h \in H^\infty(\mathbb{D}^N)$. Recall that the polydisk $\mathbb{D}^N$ is a *spectral set* for the N–tuple $T = (T_1, \dots, T_N)$ if it satisfies von Neumann's inequality, that is if $\|p(T)\| \le \|p\|_\infty$, for any polynomial p in N complex variables. Note that if the N–tuple T generates a contractive representation, then the polydisk $\mathbb{D}^N$ is a spectral set for the N–tuple.

We define the class $ACC^{(N)}(\mathfrak{H})$ composed of those N–tuples $T = (T_1, \dots, T_N)$ that generate a contractive representation of $H^\infty(\mathbb{D}^N)$. For an N–tuple to belong to $ACC^{(N)}(\mathfrak{H})$ it is sufficient that the following two conditions are satisfied, first that the polydisk is a spectral set for the N–tuple and, second, that the N–tuple is absolutely continuous in the sense of [8] (see also [5], and [7]). If $T \in ACC^{(N)}(\mathfrak{H})$, then the first condition holds. Note that the first condition is readily satisfied if $N = 2$, thanks to a celebrated result of T. Ando proving existence of dilations for any pairs of commuting contractions (see [1]) and, of course, if $N = 1$ by the standard Sz.–Nagy-Foiaş dilation theory. If $N > 2$, the first condition is essential for our techniques to work. On the other hand, if the second conditions fails, i.e., if the N–tuple of commuting contractions is not absolutely continuous, then it can be deduced from results in [5] that there exists a common nontrivial invariant subspace for the N–tuple. That is, a subspace $\mathfrak{M}$ of $\mathfrak{H}$, which is nontrivial ($\{0\} \ne \mathfrak{M} \ne \mathfrak{H}$) and invariant for each T_j ($T_j\mathfrak{M} \subset \mathfrak{M}$).

Let $T = (T_1, \dots, T_N)$ be an N–tuple of commuting operators acting on $\mathfrak{H}$. We say that T is *left invertible* if there exist an N–tuple of operators (not necessarily commuting) $A_1, \dots, A_N$ such that

$$A_1T_1 + \dots + A_NT_N = I,$$

where I denotes the identity operator on $\mathfrak{H}$. Similarly, we define *right invertibility* of an N–tuple. The *left (right) spectrum* of T, denoted $\sigma_l(T)$ ($\sigma_r(T)$), consist of those N–tuples $(\lambda_1, \dots, \lambda_N)$ of complex numbers such that $(T_1 - \lambda_1, \dots, T_N - \lambda_N)$ is not a left (right) invertible N–tuple. Note that $\sigma_r(T)^* = \sigma_l(T^*)$, where $\sigma_r(T)^*$ denotes those λ such that $\bar{\lambda} \in \sigma_r(T)$. We define the *left essential spectrum* of T by

$$\begin{aligned}\sigma_{le}(T) := \{(\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N \ &: \text{ there exist an orthonormal sequence of vectors} \\ &\{e_n\} \subset \mathfrak{H} \text{ such that } \|(T_1 - \lambda_1)e_n)\| + \dots + \|(T_N - \lambda_N)e_n)\| \to 0, \\ &\text{as } n \to \infty\}.\end{aligned}$$

We define the *right essential spectrum* of T by $\sigma_{re}(T) = \sigma_{le}(T^*)^*$.

The union of the left and right (essential) spectrum is called the Harte (essential) spectrum. These definitions, and our notation so far, coincide with those given in [4]. Let us, for the sake of simplicity, denote by σ_{He} the union of σ_{le} and σ_{re} and, similarly, denote by σ_H the union of σ_l and σ_r.

Our main result is as follows

Theorem 1. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of commuting contractions acting on $\mathfrak{H}$ having the polydisk as a spectral set. If $\sigma_H(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ then T has a common (nontrivial) invariant subspace.*

The hypothesis that $\mathbb{D}^N$ is a spectral set is readily satisfied in the case when $N = 2$, by the famous result of Ando [1] about the existence of a joint unitary dilation for pairs. Hence, we obtain the following

Theorem 2. *Let $T = (T_1, T_2)$ be a pair of commuting contractions acting on $\mathfrak{H}$. If $\sigma_H(T) \cap \mathbb{D}^2$ is dominating for $\mathbb{T}^2$ then T has a common (nontrivial) invariant subspace.*

Since T is an N-tuple of commuting contractions the sequence $T^{*n}T^n$ is a nonincreasing sequence of positive operators satisfying the equality

$$0 \le T^{*n}T^n \le 1. \tag{1}$$

We recall here that for two multiindices s, t the relation $s < t$ means that $s_j < t_j$ for $j = 1, \dots, N$. It is well known that any nonincreasing sequence of positive bounded operators has a limit in the strong operator topology which is a nonnegative bounded operator. (Due to the metrizability of the strong convergence on bounded sets of operators, we may pass from the nets of multiindices to sequences.) So, we can define the following nonnegative bounded operator on $\mathfrak{H}$:

$$B := \lim_{n \to \infty} T^{*n}T^n = \inf_n T^{*n}T^n.$$

Denote by E the spectral measure of the operator B. Since by (1) B is positive with norm less or equal 1, we have the following spectral representation

$$B = \int_0^1 t\, dE.$$

By a *completely nonisometric* N-tuple of contractions we mean an N-tuple having no nonzero common invariant subspaces on which the restriction of the N-tuple to such subspace would be isometric.

Using the properties of the operator B and its spectral measure E, we get a theorem, which says that for each fixed $x \in \mathfrak{H}$ and each integer n we can find a sequence of multiindices $k_1, \dots, k_n$ such that the vectors $T^{k_1}x, \dots, T^{k_n}x$ are "almost orthogonal".

Theorem 3. *Let $T = (T_1, \dots, T_N)$ be a completely nonisometric N-tuple of commuting contractions and let $\delta > 0$ be given. Then for a fixed $x \in \mathfrak{H}$ and an arbitrary $n \in \mathbb{N}$ we can find a finite sequence of real numbers $0 < t_1 < \cdots < t_n < t_{n+1} < 1$ and a corresponding sequence of multiintegers $k_1 < \cdots < k_n$ such that*

$$\|E([0,1] \setminus \sigma_i)T^{k_i}x\| < \delta$$

for $i = 1, \dots, n$ and $\sigma_i = [t_i, t_{i+1})$.

As the consequence we obtain the following "vanishing property"which is an essential generalization of Apostol Lemma ([2], Lemma 4.3).

Theorem 4. *Let $T = (T_1, \dots, T_N)$, $\nu > 0$ and $x \in \mathfrak{H}$ with $\|x\| \le 1$ be given. Then there exist $\delta > 0$ and $0 < r < 1$ such that for each fixed $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N$ with $|\lambda_j| > r$, $j = 1, \dots, N$, and all vectors $y \in \mathfrak{H}$ of norm one, with $\|(T_j - \lambda_j)y\| < \delta$, for $j = 1, \dots, N$, we have $|(Wx, y)| < \nu$ for any operator W in the commutant of $T_1, \dots, T_N$ such that $\|W\| \le 1$.*

The obove theorem together with others, more classical "vanishing properties"or their genaralizations leads to the following

Theorem 5. *Let $T = (T_1, \dots, T_N) \in ACC^{(N)}(\mathfrak{H})$. Then in each of the following situations T has a common (nontrivial) invariant subspace*

1) $\sigma_{le}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$.
2) $\sigma_{re}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$.
3) $\sigma_{He}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and $T_j \in C_{0\cdot}$ for $j = 1, \dots, N$.
4) $\sigma_{He}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and $T_j \in C_{\cdot 0}$ for $j = 1, \dots, N$.
5) $\sigma_{He}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and $T_j \in C_{0\cdot}, T_k \in C_{\cdot 0}$ for some j, k.

Recall that an operator $S \in C_{0\cdot}$ if $S^n x \to 0$ for $n \to \infty$, $x \in \mathfrak{H}$ and $S \in C_{\cdot 0}$ if $S^* \in C_{0\cdot}$.

Proof of Theorem 1. It is well-known that if $\lambda \in \sigma_l(T) \setminus \sigma_{le}(T)$, then λ is an eigenvalue. A similar argument works for the right spectrum and consequently also for Harte spectrum. Thus, in terms of finding common (nontrivial) invariant subspaces, we can always assume that $\sigma_l(T) = \sigma_{le}(T)$, $\sigma_r(T) = \sigma_{re}(T)$ and $\sigma_H(T) = \sigma_{He}(T)$.

If for some j the operator T_j is not in $C_{0\cdot} \cup C_{\cdot 0}$, then, by the well known result of Sz.-Nagy and C.Foiaş (see [10] or Theorem 2.2 of [3]), T_j is quasisimilar to a unitary operator. Consequently T_j has nontrivial hyperinvariant subspaces unless it is equal to the multiplication by a constant. But in the last case the problem is reduced to finding invariant subspaces for the rest of operators in the N-tuple. So, we may assume that for every j, the contraction T_j is either in $C_{0\cdot}$ or in $C_{\cdot 0}$. The rest of the proof consists of reducing to the cases of Theorem 5 through the use of the results contained in [5] to eliminate the situation when the N-tuple T is not absolutely continuous.

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Поступила в редакцию 27.09.2001