Ученые записки Таврического национального университета им. В. И. Вернадского
Серия "Математика. Механика. Информатика и Кибернетика"N 1 (2002) 136 – 141
УДК 517.95

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SINGULAR CAUCHY PROBLEMS AND PROBLEMS WITHOUT INITIAL DATA FOR NONLINEAR SYSTEMS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

Keywords: Functional-differential equations, nonsummable singularities, singular initial problems, existence and uniqueness theorems

This paper deals with a system of nonlinear functional-differential equations (FDEs) with a nonsummable singularity at infinity. We consider a singular Cauchy problem with the initial data at infinity or the problems without initial data, e.g., with the requirement of the solution boundedness. We formulate the existence and uniqueness theorems being more common than obtained in [1], [2], [3], [4].

1. Notation: $I_T = [T, \infty)$, $T \geq T_0$, T_0 is fixed; $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$, $|\cdot|$ is a norm in $\mathbf{K}^n$ or associated matrix norm in the linear space $\mathbf{L}(\mathbf{K}^n)$ of $n \times n-$matrices; $\Omega_n(a) = \{x : x \in \mathbf{K}^n, |x| \leq a\}$, where either $0 < a = a_0$ is fixed or $0 < a$ is arbitrary; $C_n(I_T)$ is the Banach space of bounded continuous functions $\xi(t)$, $\xi : I_T \to \mathbf{K}^n$, with the norm $|\xi|_C = |\xi|_{C_n(I_T)} = \sup\limits_{t \in I_T} |\xi(t)|$; $S_n(a) = \{\xi(t) : \xi \in C_n(I_T),\ |\xi|_C \leq a\ (a > 0)\}$; $L_n^\infty(I_T)$ is the Banach space of essentially bounded Lebesgue-measurable functions $\xi(t)$, $\xi : I_T \to \mathbf{K}^n$, with the norm $|\xi|_\infty = |\xi|_{L_n^\infty(I_T)} = \inf\limits_{\mu(N)=0} \sup\limits_{I_T \setminus N} |\xi(t)| =$ $\mathrm{vraisup}_{t \in I_T} |\xi(t)|$, where μ is the Lebesgue measure; $AC_n^{loc}(I_T)$ is the class of locally absolutely continuous functions $\xi(t)$, $\xi : I_T \to \mathbf{K}^n$; $L_n^{loc}(I_T)$ is the class of locally summable functions $\xi(t)$, $\xi : I_T \to \mathbf{K}^n$.

2. Subsets of x− Lipschitz functions

Let G_n be a region in $\mathbf{K}^n$ or all the space and let $\mathrm{Lip}_n = \mathrm{Lip}_n(I_{T_0} \times G_n)$ be the class of functions $f(t, x)$, $f : I_{T_0} \times G_n \to \mathbf{K}^n$, such that $f(\cdot, x)$ is continuous $\forall x \in G_n$ and in any fixed $\Omega_n(a) \subseteq G_n$ $(a > 0)$ $f(t, \cdot)$ satisfies the Lipschitz condition uniformly with respect to $t \in I_{T_0}$ with a constant $L_f = L_f(a) > 0$. We decompose these functions on four subsets: $\mathrm{Lip}_n = \mathrm{Lip}_{n,\delta_\varepsilon}(\varepsilon) \cup \mathrm{Lip}_{n,a_0} \cup \mathrm{Lip}_n(a) \cup \tilde{\mathrm{Lip}}_n$. Here the following classes are distinguished:

1) $\mathrm{Lip}_{n,\delta_\varepsilon}(\varepsilon) = \{f(t,x) : f \in \mathrm{Lip}_n(I_{T_0} \times G_n)$ and $\forall \varepsilon > 0\ \exists \delta_\varepsilon,\ T_\varepsilon,\ \delta_\varepsilon > 0,\ \Omega_n(\delta_\varepsilon) \subseteq G_n$, $T_\varepsilon \geq T_0$, such that in the region $I_{T_\varepsilon} \times \Omega_n(\delta_\varepsilon)$ we can choose $L_f = L_f(\delta_\varepsilon) = \varepsilon\}$;

2) $\mathrm{Lip}_{n,a_0} = \{f(t,x) : f \in \mathrm{Lip}_n(I_{T_0} \times \Omega_n(a_0)),\ 0 < a_0$ is fixed, with $L_f = L_f(a_0) > 0\}$;

3) $\mathrm{Lip}_n(a) = \{f(t,x) : f \in \mathrm{Lip}_n(I_{T_0} \times \mathbf{K}^n)$ and $\sup\limits_{a>0} L_f(a) = \infty$ for any choice of $L_f(a) > 0\}$;

4) $\tilde{\mathrm{Lip}}_n = \{f(t,x) : f \in \mathrm{Lip}_n(I_{T_0} \times \mathbf{K}^n)$ and $\forall a > 0$ there exist $L_f(a) > 0$ such that $\tilde{L}_f = \sup\limits_{a>0} L_f(a) < \infty\}$.

It is obvious that, in general, the intersection of the subset $\mathrm{Lip}_{n,\delta_\varepsilon}(\varepsilon)$ with the subset Lip_{n,a_0} (in the same way with the subset $\mathrm{Lip}_n(a)$ or subset $\tilde{\mathrm{Lip}}_n$) is nonempty.

3. Statement of the problems and preliminary remarks

We consider a system of n nonlinear FDEs on a semi-infinite interval in the form:

$$x' = A(t)x + M(t)(FNx)(t) + g(t) \quad \text{a.e. on } I_T. \tag{1}$$

Here, in general, the left end of I_T, $T \geq T_0$, is mobile and defined in the theorems; $x : I_T \to \mathbf{K}^n$, $g : I_{T_0} \to \mathbf{K}^n$; $A, M : I_{T_0} \to \mathbf{L}(\mathbf{K}^n)$, the entries of $A(t)$, $M(t)$, $g(t)$ are locally summable functions, i.e., belonging to the class $L_1^{loc}(I_{T_0})$; N is a local Nemytskii operator, $N : C_n(I_{T_0}) \to C_n(I_{T_0})$, such that

$$(Nx)(t) = (N_f x)(t) \equiv f(t, x(t)), \quad f \in \mathrm{Lip}_n, \quad f(t,0) \equiv 0; \tag{2}$$

$F : C_n(I_T) \to L_n^\infty(I_T)$, $(FNx)(t) = (F \circ f(\cdot, x(\cdot)))(t)$, where a mapping F, generally speaking, nonlinear, nonlocal and depending on a choice of T, satisfies conditions [4]:

$$F(0) = 0, \qquad |F(\xi) - F(\tilde{\xi})|_\infty \le |\xi - \tilde{\xi}|_C \qquad \forall \xi,\ \tilde{\xi} \in C_n(I_T). \tag{3}$$

We look for the bounded solutions of the equation (1) belonging to the class $AC_n^{loc}(I_T)$. More exactly, we consider the following problems.

Problem 1. It is required to define $x(t)$, $x \in AC_n^{loc}(I_T)$, satisfying equation (1) and restriction

$$\sup_{t \in I_T} |x(t)| \le \omega, \quad \omega > 0, \tag{4}$$

where ω is a certain finite, in general, mobile and depending on T magnitude determined in the theorems (the first problem without initial data).

Problem 2. It is required to define $x(t)$, $x \in AC_n^{loc}(I_T)$, satisfying equation (1) and the boundedness condition:

$$\sup_{t \in I_T} |x(t)| < \infty \tag{5}$$

(the second problem without initial data).

Problem 3. It is required to define $x(t)$, $x \in AC_n^{loc}(I_T)$, satisfying equation (1) and limiting condition at infinity

$$\lim_{t \to \infty} x(t) = 0 \tag{6}$$

(singular Cauchy problem with the initial data at infinity).

3.1. Special classes of the mapping F

The following partial classes of the mapping F are practically important.

Assumption 1. The mapping F, $F : C_n(I_{\tilde{T}}) \to L_n^\infty(I_{\tilde{T}})$ ($\tilde{T} \ge T_0$), is a singular Volterra operator (SVO), i.e., $\forall T \ge \tilde{T}$ and $\forall \xi_1, \xi_2 \in C_n(I_{\tilde{T}})$ from an equality $\xi_1(t) = \xi_2(t)$ on I_T follows

$$(F\xi_1)(t) = (F\xi_2)(t) \quad \text{a.e. on } I_T.$$

Assumption 2. The mapping F, $F : C_n(I_{T_0}) \to L_n^\infty(I_{T_0})$, is a local Nemytskii operator, i.e.,

$$(F\xi)(t) \equiv (N_\varphi \xi)(t) \equiv \varphi(t, \xi(t)),$$

where $\xi \in C_n(I_{T_0})$, $\varphi : C_n(I_{T_0}) \to L_n^\infty(I_{T_0})$.

Remark 3. For the theory of FDEs defined on a finite interval and containing (non-)Volterra operators, see, e.g., [5].

Remark 4. If Assumption 2 is satisfied, then (1) is a system of generalized ordinary differential equations (ODEs); if F is an embedding $C_n(I_{T_0})$ into $L_n^\infty(I_{T_0})$, i.e., $F(\xi) \equiv \xi\ \forall \xi \in C_n(I_{T_0})$, and the entries of $A(t)$, $M(t)$, $g(t)$ are piecewise continuous functions on I_{T_0}, then (1) is simply a system of ODEs. If $(F\xi)(t) = \xi(h(t))$, $h : I_T \to I_T$, then (1) is a system of differential-delay equations; if F is an integral operator, then (1) is a system of integral-differential equations, etc.

3.2. On a singularity at infinity and the Caratheodory-type conditions

We want to adapt the contraction mapping principle to the operator equation

$$x(t) = (V(x))(t), \quad t \ge T. \tag{7}$$

where $V : C_n(I_T) \to C_n(I_T)$, and functional-integral equation (7) should be equivalent to Problem 1, either Problem 2 or Problem 3 respectively. The operator V construction, in particular, depends on a singular point type at infinity.

Definition. We say that the equation (1) has a summable singularity at infinity if, and only if, the inequalities

$$I_A = \int_{T_0}^{\infty} |A(t)|dt < \infty, \qquad I_M = \int_{T_0}^{\infty} |M(t)|dt < \infty, \tag{8}$$

$$I_g = \int_{T_0}^{\infty} |g(t)|dt < \infty \tag{9}$$

are valid, otherwise a singularity at infinity is said to be a nonsummable one.

If the relations (8), (9) are fulfilled, then we define

$$(V(x))(t) = -\int_{t}^{\infty} [A(s)x(s) + M(s)(FNx)(s) + g(s)]ds, \quad t \geq T_0,$$

and it is easily to reformulate the Caratheodory-type theorems for the indicated problems to the equation (1) (for the Caratheodory-type theorems to generalized ODEs, see, e.g., [6]).

Remark 5. In general, here and in what follows the integration is in the Lebesgue sense. For the equation (1), if it is possible to use the improper integrals in the Riemann sense, then we suppose that

$$\tilde{I}_g = \left|\int_{T_0}^{\infty} g(t)dt\right| < \infty, \tag{10}$$

where the integral standing under the modulus can be convergent conditionally. Then the relations (8), (10) correspond to the Kudryavtsev-type conditions for ODEs [7].

The goal of this paper is to consider Problems 1, 2, 3 to the equation (1) with a nonsummable singularity at infinity. As it has been demonstrated by Chechik on the example for ODE (see [8]), the Caratheodory-type conditions, i.e., the restrictions to a growth of given functions with respect to t, generally speaking, cannot provide an existence and uniqueness of a solution to singular Cauchy problem with the initial data at a nonsummable singularity. For singular Cauchy problems to nonlinear systems of ODEs with the initial data at a regular (irregular) singular point, see, e.g., [9].

4. The existence and uniqueness theorems

Let $\Phi_A(t)$ be a fundamental matrix for a system

$$x' = A(t)x \quad \text{a.e. on } I_{T_0}. \tag{11}$$

We denote by $U_A(t,s)$ the Cauchy matrix, $U_A(t,s) = \Phi_A(t)\Phi_A^{-1}(s)$, and introduce the auxiliary quantities

$$J_M(t) = \int_{t}^{\infty} |U_A(t,s)M(s)|ds, \quad J_g(t) = \int_{t}^{\infty} |U_A(t,s)g(s)|ds, \quad t \geq T_0, \tag{12}$$

$$\hat{J}_M(T) = \sup_{t\in I_T} J_M(t), \quad \hat{J}_g(T) = \sup_{t\in I_T} J_g(t), \quad T \geq T_0, \tag{13}$$

and suppose that

$$\hat{J}_M(T_0) < \infty, \qquad \hat{J}_g(T_0) < \infty. \tag{14}$$

Remark 6. For the equation (1), if we can use the improper integrals in the Riemann sense, then we introduce the magnitude $J_g(t)$ by the formula $J_g(t) = |\int_t^\infty U_A(t,s)g(s)ds|$, $t \geq T_0$, where the integral standing under the modulus can be convergent conditionally (compare with Remark 5).

For $f(t,x)$, let the requirements (2) be valid and let $q, \omega, \tilde{T}$, $0 < q < 1$, $\omega > 0$, $\tilde{T} \geq T_0$, and the values (12), (13) be such that the relations

$$\hat{J}_M(\tilde{T}) = \sup_{t \in I_{\tilde{T}}} J_M(t) \leq q/L_f, \tag{15}$$

$$\hat{J}_g(\tilde{T}) = \sup_{t \in I_{\tilde{T}}} J_g(t) \leq \omega(1-q) \tag{16}$$

hold where L_f is the Lipschitz constant and, in addition, let a choice of ω, $\tilde{T}$, L_f be subjected to the following conditions:

(i) if $f \in \mathrm{Lip}_{n,\delta_\varepsilon}(\varepsilon)$, then we put $L_f = \varepsilon$, $\omega = \delta_\varepsilon$, $\tilde{T} = T_\varepsilon$, and the relation (15) holds for a suitable choice of $\varepsilon > 0$; if the inequality (16) is not valid for $\omega = \delta_\varepsilon$, $\tilde{T} = T_\varepsilon$, then we suppose that

$$\lim_{t \to \infty} J_g(t) = 0, \tag{17}$$

so that the relation (16) holds for a suitable choice of $\tilde{T} > T_\varepsilon$;

(ii) if $f \in \mathrm{Lip}_{n,a_0} \setminus \mathrm{Lip}_{n,\delta_\varepsilon}(\varepsilon)$, then we put $L_f = L_f(a_0)$ and fix $q, \omega, \tilde{T}$, $0 < q < 1$, $0 < \omega \leq a_0$, $\tilde{T} \geq T_0$; if for these values the inequality (15) is not valid, then we assume that

$$\lim_{t \to \infty} J_M(t) = 0, \tag{18}$$

so that (15) can be satisfied due to a suitable choice of $\tilde{T}$; in addition, if (16) is not valid, then we assume that (17) is fulfilled to choose a new $\tilde{T} \in I_{T0}$;

(iii) if $f \in \mathrm{Lip}_n(a) \cup \widetilde{\mathrm{Lip}}_n$, then we fix $q: 0 < q < 1$, and ω :

$$\omega \geq \omega_q = \hat{J}_g(T_0)/(1-q), \tag{19}$$

and put $L_f = L_f(\omega)$ or $L_f = \tilde{L}_f$ respectively; due to (19) the relation (16) holds $\forall \tilde{T} \geq T_0$, but if for fixed $L_f, \tilde{T}$ the inequality (15) is not satisfied, then we introduce the requirement (18) to choose a new $\tilde{T} \in I_{T0}$.

Let the indicated requirements be fulfilled. Let us choose $T \geq \tilde{T}$ and take in $C_n(I_T)$ a closed ball by the radius ω: $S_n(\omega) = \{x(t) : x \in C_n(I_T), |x|_C \leq \omega\}$. On this ball, we consider the mapping V, V: $C_n(I_T) \to C_n(I_T)$, defined as follows

$$(V(x))(t) = -\int_t^\infty U_A(t,s)[M(s)(FNx)(s) + g(s)]ds, \quad t \geq T. \tag{20}$$

Theorem 1. *Let $A(t), M(t), f(t,x), g(t)$ be such that the requirements (2) and (14) are fulfilled and let for a chosen $q, 0 < q < 1$, the values ω and T be defined as above.*

Then for any mapping F satisfying conditions (3) there exists a unique fixed point $\hat{x}$, $\hat{x} \in S_n(\omega)$, of the mapping V; it can be specified as the limit

$$\hat{x} = \lim_{k \to \infty} V^k(x_0) \tag{21}$$

for any starting point x_0, $|x_0|_C \leq \omega$, moreover, for the rate of convergence we have the estimate

$$|V^k(x_0) - \hat{x}|_C \leq [q^k/(1-q)]|V(x_0) - x_0|_C,$$

and the following estimate holds:

$$|\hat{x}|_C \le \hat{J}_g(T)/(1-q), \tag{22}$$

and, in addition, if Assumption 1 is valid, i.e., F is SVO, then (22) holds $\forall T \ge \tilde{T}$, that, in particular, implies a realization of limiting condition (6) if (17) is valid. Besides a global convergence of successive approximations to $\hat{x}$, i.e., $\forall x_0 \in C_n(I_T)$, occurs in the following two cases: 1) $f \in \tilde{\mathrm{Lip}}_n$; 2) $f \in \mathrm{Lip}_n(a)$, F is SVO, i.e., Assumption 1 is valid, and limiting condition (18) is fulfilled; in this case a choice of T depends on a value of x_0 determining in turn a choice of ω, $\omega \ge |x_0|_C$, and $L_f = L_f(\omega) > 0$.

Corollary. *Let the hypothesis of Theorem 1 be satisfied. Then the constructed function $\hat{x}(t)$, $\hat{x} \in AC_n^{loc}(I_T)$, is a solution of Problem 1; is F is SVO and limiting condition (17) is satisfied, then $\hat{x}$ is a solution of Problem 3; if Assumsion 2 is valid, i.e., F is a local Nemytskii operator, then : if $f \in \tilde{\mathrm{Lip}}_n$ then $\hat{x}$ exists in the large (for all $t \in [T_0, \infty)$) while if $f \in \mathrm{Lip}_n(a)$ then $\hat{x}$ is uniquely extendible to the left as long as it remains bounded (at least to $\tilde{T}_0 \ge T_0$ such that $\hat{J}_M(\tilde{T}_0) \le q_0/L_f(\omega_{q_0})$ where $q_0 : q_0/L_f(\omega_{q_0}) = \sup\limits_{0<q<1} \{q/L_f(\omega_q)\}$).*

Theorem 2. *Let all nontrivial solutions to the equation (11) be unbounded as $t \to \infty$ and let otherwise the hypothesis of Theorem 1 be satisfied.*

Then: 1) for any mapping F satisfying conditions (3), Problem 1 is equivalent, on the function class $AC_n^{loc}(I_T)$, to the operator equation (7) where V is defined by (20) so that Problem 1 has a unique solution $\hat{x}$ defined by Theorem 1; 2) if Assumption 1 is valid, i.e., F is SVO, then previous statement holds $\forall T \ge \tilde{T}$, and, in addition, if the limiting condition (17) is satisfied. then $\hat{x}$ is a unique solution of Problem 3 (as a singular Cauchy problem); 3) if $f \in \tilde{\mathrm{Lip}}_n$ ($f \in \mathrm{Lip}_n(a)$, F is SVO and (18) is true) then $\hat{x}$ is a unique solution of Problem 2, i.e., it is a unique bounded solution to the equation (1).

Theorem 3. *Let no solution of the equation (11) be tending to zero as $t \to \infty$ other than $x(t) \equiv 0$ and let the limiting condition (17) be valid; let otherwise the hypothesis of Theorem 1 be satisfied.*

Then for any mapping F satisfying conditions (3) and Assumption 1, Problem 3 is equivalent to functional-integral equation (7) where V is defined by (20), so that Problem 3 has a unique solution $\hat{x}$ defined by Theorem 1.

Remark 7. For [4], the existence and uniqueness theorem to a singular Cauchy problem is a particular case of Theorem 3. If $A(t) \equiv 0$, then Theorem 3 turns into the Caratheodory- or Kudryavtsev-type theorem respectively.

5. Model example with a linear non-Volterra operator

Let us consider an example generalizing one suggested by referee E.I. Bravyi (see [3]). We consider FDE

$$x' = ax/t + bx(1)/t + d/t^3, \quad 1 \le t < \infty, \tag{23}$$

where a, b, d are parameters, $a + b \ne 0$. The general solution of the equation (23) is given as follows: $x(t) = ct^a - d/[t^2(a+2)] - b[c(a+2) - d]/[(a+b)(a+2)]$ $(a \ne 0 \wedge a \ne -2)$, $x(t) = c + b(c - d/2)\ln t - d/(2t^2)$ $(a = 0)$, $x(t) = c/t^2 + (d/t^2)\ln t + cb/(2-b)$ $(a = -2)$, where c is an arbitrary constant. For $a \ge 0$, there exists a unique solution bounded as $t \to \infty$:

$$x(t) = d[b/(a+b) - 1/t^2]/(a+2), \quad a \ge 0. \tag{24}$$

In our notation we obtain $n = 1$, $A(t) = a/t$, $M(t) = b/t$, $g(t) = d/t^3$, $f(t,x) \equiv x$, $f \in \tilde{\mathrm{Lip}}_1$, $(FNx)(t) \equiv (Fx)(t) \equiv x(1)$, $L_f = 1$. For $a > 0$, we define $J_M(t) = |b| \int_t^\infty (t^a/s^{a+1})ds \equiv |b|/a$,

$J_g(t) = |d| \int_t^\infty (t^a/s^{a+3})ds \equiv |d|/[t^2(a+2)]$. Accoding to the theorems of Section 4, we fix q, $0 < q < 1$, and suppose that

$$\hat{J}_M(1) = |b|/a \le q, \quad \hat{J}_g(1) = |d|/(a+2) \le \omega(1-q). \tag{25}$$

To satisfy (25) for $\hat{J}_g(1)$, we take $\omega \ge \omega_q = |d|/[(a+2)(1-q)]$. In $S_1(\omega)$ we consider the equation

$$x(t) = -\int_t^\infty (t/s)^a[bx(1)/s + d/s^3]ds, \quad t \ge 1, \tag{26}$$

which has an exact solution, namely, from (26) we have $x(1) = -bx(1)/a - d/(a+2)$, so that once more from (26) we obtain

$$x(t) = -\int_t^\infty (t/s)^a\{d/s^3 - abd/[s(a+2)(a+b)]\}ds \equiv d[b/(a+b) - 1/t^2]/(a+2), \tag{27}$$

that is the same as (24). For the exact formula (24), we have the estimate $|x(t)| \le |d| \max\{a, |b|\}/[|a+b|(a+2)]$, and, accoding to the theorems of Section 4, we obtain the estimate $|\hat{x}(t)| \le |d|/[(a+2)(1-q)]$, if only $|b|/a \le q$. It is easily to check it for the exact solution.

It should be noted that: 1) although $J_g(t) \to 0$ as $t \to \infty$, $x(t)$ does not tend to zero as $t \to \infty$, but it is not a contradiction because F is not a Volterra operator; 2) for $a = 0$, we cannot use our theorems because $J_M(1) = |b| \int_1^\infty (1/s)ds = \infty$.

For a history of matter, other examples and applications, see, e.g., [1], [2], [3], [4], [9].

This work was supported by RFBR, poject N 02-01-00050.

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Поступила в редакцию 14.12.2001