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S. Hassi, L. L. Oridoroga

THEOREM OF COMPLETENESS FOR A DIRAC TYPE OPERATOR WITH GENERALIZED λ-DEPENDING BOUNDARY CONDITIONS

A completeness theorem is proved involving a system of integro-differential equations with some λ-depending boundary conditions.

1. It is well known [5] that the system of eigenfunctions and associate functions (SEAF) of the Sturm – Liouville problem

$$-y'' + q(x)y = \lambda^2 y, \tag{1}$$

$$y'(0) - h_0 y(0) = y'(1) - h_1 y(1) = 0, \tag{2}$$

is complete in $L_2[0,1]$ for arbitrary complex valued potential $q \in L_1[0,1]$ and $h_0, h_1 \in \mathbb{C}$. A similar result is also known for arbitrary nondegenerate boundary conditions (see [5]).

A completeness result for a boundary value problem of arbitrary order differential equations

$$y^{(n)} + \sum_{j=0}^{n-2} q_j(x) y = \lambda^n y, \tag{3}$$

subject to spliting boundary conditions, has been announced by M.V. Keldysh [1] and it was first proved by A.A. Shkalikov [9].

In [4] the above mentioned results from [5] have been generalized to the case of first order systems with arbitrary boundary conditions (not depending on a spectral parameter).

In [10] and [11] the completeness results for the problem (1), (2) have been generalized to the case of nonlinear λ-depending boundary conditions of the form

$$\begin{cases} P_{11}(\lambda)y(0) + P_{12}(\lambda)y'(0) = 0 \\ P_{21}(\lambda)y^2(\frac{1}{2}) + P_{22}(\lambda)y(\frac{1}{2})y'(\frac{1}{2}) + P_{23}(\lambda){y'}^2(\frac{1}{2}) = 0 \end{cases} \tag{4}$$

and of the form

$$\begin{cases} P_{10}(\lambda)y^2(0) + P_{11}(\lambda){y'}^2(0) + P_{12}(\lambda)y^2(\frac{1}{3}) + P_{13}(\lambda){y'}^2(\frac{1}{3}) \\ \quad + P_{14}(\lambda)y(0)y'(0) + P_{15}(\lambda)y(0)y(\frac{1}{3}) + P_{16}(\lambda)y(0)y'(\frac{1}{3}) \\ \quad + P_{17}(\lambda)y'(0)y(\frac{1}{3}) + P_{18}(\lambda)y'(0)y'(\frac{1}{3}) + P_{19}(\lambda)y(\frac{1}{3})y'(\frac{1}{3}) = 0, \\ \\ P_{20}(\lambda)y^2(0) + P_{21}(\lambda){y'}^2(0) + P_{22}(\lambda)y^2(\frac{1}{3}) + P_{23}(\lambda){y'}^2(\frac{1}{3}) \\ \quad + P_{24}(\lambda)y(0)y'(0) + P_{25}(\lambda)y(0)y(\frac{1}{3}) + P_{26}(\lambda)y(0)y'(\frac{1}{3}) \\ \quad + P_{27}(\lambda)y'(0)y(\frac{1}{3}) + P_{28}(\lambda)y'(0)y'(\frac{1}{3}) + P_{29}(\lambda)y(\frac{1}{3})y'(\frac{1}{3}) = 0. \end{cases} \tag{5}$$

Moreover, in [7] and [8] analogous results were obtained for a system with a pair of spliting λ-depending boundary conditions similar to the conditions (4) and (5).

In the present paper the completeness result is generalized to the case of first order systems consisting of two integro-differential equations with arbitrary linear or quadratic λ-depending boundary conditions. More precisely, let $B = \mathrm{diag}(a^{-1}, b^{-1})$ be a 2×2 diagonal matrix with

$a < 0 < b$. Consider in $L_2[0,1] \oplus L_2[0,1]$ a boundary value problem for the first order system of ordinary integro-differential equations of the form

$$\frac{1}{i}By' + Q(x)y + \int_0^x M(x,t)y(t)\,dt = \lambda y. \tag{6}$$

Here

$$Q(x,t) = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}, \quad M(x,t) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

where it is assumed that $q_j \in L_1[0,1]$ and $M_{ij} \in L_\infty(\Omega)$, $\Omega = \{0 \le t \le x \le 1\}$, $i,j = 1,2$. Two types of λ-depending boundary conditions will be treated. Namely:

(i) arbitrary linear conditions of the form

$$\begin{cases} P_{11}(\lambda)y_1(0) + P_{12}(\lambda)y_2(0) + P_{13}(\lambda)y_1(1) + P_{14}(\lambda)y_2(1) = 0 \\ P_{21}(\lambda)y_1(0) + P_{22}(\lambda)y_2(0) + P_{23}(\lambda)y_1(1) + P_{24}(\lambda)y_2(1) = 0 \end{cases} \tag{7}$$

or:

(ii) arbitrary quadratic conditions of the form

$$\begin{cases} P_{10}(\lambda)y_1^2(0) + P_{11}(\lambda)y_2^2(0) + P_{12}(\lambda)y_1^2(\frac{1}{2}) + P_{13}(\lambda)y_2^2(\frac{1}{2}) \\ \quad + P_{14}(\lambda)y_1(0)y_2(0) + P_{15}(\lambda)y_1(0)y_1(\frac{1}{2}) + P_{16}(\lambda)y_1(0)y_2(\frac{1}{2}) \\ \quad + P_{17}(\lambda)y_2(0)y_1(\frac{1}{2}) + P_{18}(\lambda)y_2(0)y_2(\frac{1}{2}) + P_{19}(\lambda)y_1(\frac{1}{2})y_2(\frac{1}{2}) = 0 \\ \\ P_{20}(\lambda)y_1^2(0) + P_{21}(\lambda)y_2^2(0) + P_{22}(\lambda)y_1^2(\frac{1}{2}) + P_{23}(\lambda)y_2^2(\frac{1}{2}) \\ \quad + P_{24}(\lambda)y_1(0)y_2(0) + P_{25}(\lambda)y_1(0)y_1(\frac{1}{2}) + P_{26}(\lambda)y_1(0)y_2(\frac{1}{2}) \\ \quad + P_{27}(\lambda)y_2(0)y_1(\frac{1}{2}) + P_{28}(\lambda)y_2(0)y_2(\frac{1}{2}) + P_{29}(\lambda)y_1(\frac{1}{2})y_2(\frac{1}{2}) = 0 \end{cases} \tag{8}$$

where $P_{ij}(\lambda)$ are polynomials.

Below some sufficient conditions for the completeness of the SEAF of the problems (6), (7) and (6), (8) in $L^2[0,1] \oplus L^2[0,1]$ are established. A starting point is to estimate the growth of the solution of the Cauchy problem for the system (6) with special initial conditions. Indeed, let

$$\vec{\varphi}_0(x;\lambda) = \begin{pmatrix} \varphi_{01}(x;\lambda) \\ \varphi_{02}(x;\lambda) \end{pmatrix} \quad \text{and} \quad \vec{\psi}_0(x;\lambda) = \begin{pmatrix} \psi_{01}(x;\lambda) \\ \psi_{02}(x;\lambda) \end{pmatrix} \tag{9}$$

be the solutions of the Cauchy problem for the system (6) with the initial conditions

$$\varphi_{01}(0;\lambda) = \psi_{02}(0;\lambda) = 1 \quad \text{and} \quad \varphi_{02}(0;\lambda) = \psi_{01}(0;\lambda) = 0, \tag{10}$$

respectively. The next lemma gives some estimates for the growth of $\vec{\varphi_i}(x;\lambda)$ and $\vec{\psi_i}(x;\lambda)$, $i = 1,2$.

Lemma 1. *The functions $\vec{\varphi}_i(x;\lambda)$ and $\vec{\psi}_i(x;\lambda)$, $i = 0,1$, satisfy the estimates*

$$\begin{aligned} \varphi_{01}(x;\lambda) &= (1 + O(\tfrac{1}{\Im\lambda}))\exp(a\lambda ix), & \varphi_{02}(x;\lambda) &= O(\tfrac{1}{\Im\lambda})\exp(a\lambda ix); \\ \psi_{01}(x;\lambda) &= O(\tfrac{1}{\Im\lambda})\exp(a\lambda ix), & \psi_{02}(x;\lambda) &= O(\tfrac{1}{\Im\lambda})\exp(a\lambda ix); \end{aligned} \tag{11}$$

when $\lambda \in \mathbb{C}^+$, and the estimates

$$\begin{aligned} \varphi_{01}(x;\lambda) &= O(\tfrac{1}{\Im\lambda})\exp(b\lambda ix), & \varphi_{02}(x;\lambda) &= O(\tfrac{1}{\Im\lambda})\exp(b\lambda ix); \\ \psi_{01}(x;\lambda) &= O(\tfrac{1}{\Im\lambda})\exp(b\lambda ix), & \psi_{02}(x;\lambda) &= (1 + O(\tfrac{1}{\Im\lambda}))\exp(b\lambda ix); \end{aligned} \tag{12}$$

when $\lambda \in \mathbb{C}^-$.

The proof is based on the existence of a triangular trasformation operator for the equation (6) constructed in [3].

The completeness result can now be stated as follows.

Theorem 1. *Let P_{ij} (i=1,2 ; j=1,2,3,4) be polynomials, let the rank of the polynomial matrix*

$$P(\lambda) = \begin{pmatrix} P_{11}(\lambda) & P_{12}(\lambda) & P_{13}(\lambda) & P_{14}(\lambda) \\ P_{21}(\lambda) & P_{22}(\lambda) & P_{23}(\lambda) & P_{24}(\lambda) \end{pmatrix} \tag{13}$$

be equal to 2 for all $\lambda \in \mathbb{C}$, and let

$$\deg J_{14} = \deg J_{32} \geq \max\{\deg J_{13}, \deg J_{42}, M\}, \tag{14}$$

where $M = \max\{\deg P_{ij} : \ i \in \{1,2\};\ j \in \{1,2,3,4\}\}$ and

$$J_{ij} = \det \begin{pmatrix} P_{1i} & P_{1j} \\ P_{2i} & P_{2j} \end{pmatrix}, \quad i,j = \{1,2,3,4\}. \tag{15}$$

Then the SEAF of the problem (6), (7) *is complete in* $L^2[0,1] \oplus L^2[0,1]$.

Moreover, let the set Φ, which consists of $N := \deg J_{14} - M$ eigenfunctions and associate functions, satisfy the following condition:

If Φ contains either an eigenfunction or an associate function corresponding to an eigenvalue λ_k, then it also contains all the associate functions of higher order corresponding to the same eigenvalue.

Then the SEAF of the problem (6), (7) *without the set Φ is also complete in the space* $L^2[0,1] \oplus L^2[0,1]$.

Proof. First observe that λ is an eigenvalue if and only if the so-called the characteristic function $\chi(\lambda)$ of the problem (6), (7) satisfies

$$\chi(\lambda) := \det \begin{pmatrix} Q_{11}(\lambda) & Q_{12}(\lambda) \\ Q_{21}(\lambda) & Q_{22}(\lambda) \end{pmatrix} = 0. \tag{16}$$

Here

$$\begin{aligned} Q_{11}(\lambda) &= P_{11}(\lambda) + P_{13}(\lambda)\varphi_{01}(1;\lambda) + P_{14}(\lambda)\varphi_{02}(1;\lambda), \\ Q_{12}(\lambda) &= P_{12}(\lambda) + P_{13}(\lambda)\psi_{01}(1;\lambda) + P_{14}(\lambda)\psi_{02}(1;\lambda), \\ Q_{21}(\lambda) &= P_{21}(\lambda) + P_{23}(\lambda)\varphi_{01}(1;\lambda) + P_{24}(\lambda)\varphi_{02}(1;\lambda), \\ Q_{22}(\lambda) &= P_{22}(\lambda) + P_{23}(\lambda)\psi_{01}(1;\lambda) + P_{24}(\lambda)\psi_{02}(1;\lambda). \end{aligned} \tag{17}$$

If in addition λ_0 is a root of $\chi(\lambda)$ of the order p, then the operator determined by (6), (7) has precisely p eigenfunctions and associate functions corresponding to λ_0. Introduce the functions

$$\begin{aligned} \vec{\omega_1}(x;\lambda) = {} & (P_{12}(\lambda) + (P_{13}(\lambda)\psi_{01}(1;\lambda) + (P_{14}(\lambda)\psi_{02}(1;\lambda))\varphi_0(x;\lambda) \\ & - (P_{11}(\lambda) + (P_{13}(\lambda)\varphi_{01}(1;\lambda) + (P_{14}(\lambda)\varphi_{02}(1;\lambda))\psi_0(x;\lambda) \end{aligned} \tag{18}$$

and

$$\begin{aligned} \vec{\omega_2}(x;\lambda) = {} & (P_{22}(\lambda) + (P_{23}(\lambda)\psi_{01}(1;\lambda) + (P_{24}(\lambda)\psi_{02}(1;\lambda))\varphi_0(x;\lambda) \\ & - (P_{21}(\lambda) + (P_{23}(\lambda)\varphi_{01}(1;\lambda) + (P_{24}(\lambda)\varphi_{02}(1;\lambda))\psi_0(x;\lambda). \end{aligned} \tag{19}$$

Then for all λ the function $\vec{\omega_j}(x;\lambda)$ is a solution of the first equation in (7). If λ_0 is a root of $\chi(\lambda)$ of order p then $\vec{\omega_j}(x;\lambda_0)$ is a solution of the second equation in (7) too. Moreover, since the function

$$P_{21}\omega_{j1}(0,\lambda) + P_{22}\omega_{j2}(0,\lambda) + P_{23}\omega_{j1}(1,\lambda) + P_{24}\omega_{j2}(1,\lambda) \tag{20}$$

is equal to $-\chi(\lambda)$, all nonzero functions

$$\left.\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\omega_j(x;\lambda)\right|_{\lambda=\lambda_0}, \qquad \text{where } 1 \leq k < p, \quad j = 1,2, \tag{21}$$

are eigenfunctions and associate functions corresponding to the eigenvalue λ_0.

Now suppose that the SEAF of the problem (6), (7) without the set Φ is not complete in the space $L^2[0,1]\oplus L^2[0,1]$. Then there exists a nonzero vector function $\vec{f}(x)=(f_1(x),f_2(x))^\top$, which is orthogonal to the SEAF of the problem (6), (7) (possibly, excluding functions from the set Φ). Define

$$\widetilde{F_j}(\lambda):=\left\langle \vec{\omega}_j(x;\lambda),\vec{f}(x)\right\rangle=\int_0^1(\omega_{j1}(x;\lambda)\overline{f_1(x)}+\omega_{j2}(x;\lambda)\overline{f_2(x)})\,dx. \tag{22}$$

Clearly, $\widetilde{F_j}(\lambda)$ is an entire function. If λ_s is an eigenvalue of multiplicity p and the set Φ contains neither an eigenfunction nor an associate function corresponding to λ_s, then λ_s is a root of $\widetilde{F_j}(\lambda)$, $j=1,2$, of order p. If Φ contains k eigenfunctions or associate functions corresponding to the eigenvalue λ_s, then λ_s is a root of $\widetilde{F_j}(\lambda)$, $j=1,2$, of order greater than or equal to $p-k$.

Denote by Λ the set of all eigenvalues of the problem (6), (7), such that the corresponding eigenfunctions (or associate functions) belong to the set Φ. For each $\lambda_s\in\Lambda$ denote by p_s the number of eigenfunctions and associate functions in Φ corresponding to λ_s. Define

$$\Pi(\lambda)=\prod_{\lambda_s\in\Lambda}(\lambda-\lambda_s)^{p_s}. \tag{23}$$

Let λ_k be an eigenvalue of problem (6), (7) of multiplicity p. Then λ_k is a zero of the product $\Pi(\lambda)\widetilde{F}_j(\lambda)$ of order p. Consequently, the functions

$$F_j(\lambda)=\frac{\Pi(\lambda)\widetilde{F_j}(\lambda)}{\chi(\lambda)} \tag{24}$$

are entire.

Next an estimate for these functions will be derived.

One can rewrite $\chi(\lambda)$ as follows

$$\begin{aligned}\chi(\lambda)&=J_{12}+J_{13}\psi_{01}(1;\lambda)+J_{14}\psi_{02}(1;\lambda)+J_{32}\varphi_{01}(1;\lambda)\\&\quad+J_{42}\varphi_{02}(1;\lambda)+J_{34}\det\begin{pmatrix}\varphi_{01}(1;\lambda)&\psi_{01}(1;\lambda)\\\varphi_{02}(1;\lambda)&\psi_{02}(1;\lambda)\end{pmatrix}\\&=J_{12}+J_{13}\psi_{01}(1;\lambda)+J_{14}\psi_{02}(1;\lambda)+J_{32}\varphi_{01}(1;\lambda)\\&\quad+J_{42}\varphi_{02}(1;\lambda)+J_{34}\left(1+O\left(\tfrac{1}{\Im\lambda}\right)\right)\exp((a+b)\lambda i).\end{aligned}\tag{25}$$

Then one obtains from (11), (12), and the assumption (14) the following estimates for $\chi(\lambda)$:

$$\chi(\lambda)=(1+O(\tfrac{1}{\Im\lambda}))J_{32}\exp(a\lambda i),\quad \lambda\in\mathbb{C}^+; \tag{26}$$

$$\chi(\lambda)=(1+O(\tfrac{1}{\Im\lambda}))J_{14}\exp(b\lambda i),\quad \lambda\in\mathbb{C}^-. \tag{27}$$

The definition of $\vec{\omega}_j(x;\lambda)$ (cf. (18) and (19)) shows that

$$\begin{aligned}\vec{\omega}_j(x;\lambda)&=-P_{j1}(\lambda)\psi_0(x;\lambda)+P_{j2}(\lambda)\varphi_0(x;\lambda)\\&\quad+\left(1+O\left(\tfrac{1}{\Im\lambda}\right)\right)\exp((a+b)\lambda i)(-P_{j3}(\lambda)\psi_1(x;\lambda)+P_{j4}(\lambda)\varphi_1(x;\lambda)).\end{aligned}\tag{28}$$

If $\lambda\in\mathbb{C}^+$, then (28) and the estimate (11) imply

$$\begin{aligned}\omega_{jk}(x;\lambda)&=(O(P_{j1}(\lambda)+O(P_{j2}(\lambda)))\exp(a\lambda ix)\\&\quad+(O(P_{j3}(\lambda)+O(P_{j4}(\lambda)))\exp(a\lambda i)\exp(b\lambda ix).\end{aligned}\tag{29}$$

It follows from

$$\int_0^1 |f_k(x)\exp(a\lambda ix)|\,dx = O\left(\frac{\exp(a\lambda i)}{\sqrt{|\Im\lambda|}}\right), \quad \int_0^1 |f_k(x)\exp(b\lambda ix)|\,dx = O\left(\frac{1}{\sqrt{|\Im\lambda|}}\right),$$

that there exists a constant $c_1 > 0$, such that, for $\Im\lambda > c_1$,

$$\widetilde{F}_j(\lambda) = O\left(\max|P_{jk}|(\lambda)\frac{\exp(a\lambda i)}{\sqrt{|\Im\lambda|}}\right) = O\left(\frac{\lambda^M\exp(a\lambda i)}{\sqrt{|\Im\lambda|}}\right). \tag{30}$$

Similarly, there exists a constant $c_2 < 0$, such that, for $\Im\lambda < c_2$,

$$\widetilde{F}_j(\lambda) = O\left(\frac{\lambda^M\exp(a\lambda i)}{\sqrt{|\Im\lambda|}}\right). \tag{31}$$

From (26), (27), (30), and (31) one obtains finally the estimate

$$F_j(\lambda) = O\left(\frac{1}{\sqrt{|\Im\lambda|}}\right), \quad |\Im\lambda| > c. \tag{32}$$

According to Phragmen–Lindelöf theorem for a strip $F_j(\lambda) \equiv 0$. Consequently, $\widetilde{F}_j(\lambda) \equiv 0$, i.e. $\vec{f}(x)$ is orthogonal to $\vec{\omega}_1(x;\lambda)$ and $\vec{\omega}_2(x;\lambda)$ for all λ. However, the functions $\vec{\omega}_1(x;\lambda)$ and $\vec{\omega}_2(x;\lambda)$ for all λ form a fundamental system of solutions of the equation (6) if λ is not an eigenvalue. Since the set of eigenvalues coincides with the set of all roots of $\chi(\lambda)$, this set is discrete. This implies that $\vec{f}(x)$ is orthogonal to all solutions of the equation (6), so that $\vec{f}(x) \equiv 0$.

Therefore, there is no nontrivial function $\vec{f}(x)$ orthogonal to the SEAF of the problem (6)–(7) (maybe without the set Φ). □

Theorem 2. *Let P_{ij} $(i=1,2\ ;\ j=0,1,\ldots,9)$ be polynomials, let the rank of the matrix*

$$\begin{pmatrix} P_{10}(\lambda) & P_{11}(\lambda) & \ldots & P_{19}(\lambda) \\ P_{20}(\lambda) & P_{21}(\lambda) & \ldots & P_{29}(\lambda) \end{pmatrix} \tag{33}$$

be equal to 2 for all $\lambda \in \mathbb{C}$, and let

$$\deg J_{03} = \deg J_{12} = M, \tag{34}$$

where

$$J_{ij} = \det\begin{pmatrix} P_{1i} & P_{1j} \\ P_{2i} & P_{2j} \end{pmatrix}, \quad i,j = 0,1,\ldots,9, \tag{35}$$

and $M = \max\{\deg P_{ij} : i \in \{1,2\};\ j \in \{0,1,\ldots,9\}\}$. Then the SEAF of problem (6), (8) is complete in $L^2[0,1] \oplus L^2[0,1]$.

Moreover, let the set Φ, which consists of M eigenfunctions and associate functions, satisfy the following condition:

If Φ contains either an eigenfunction or an associate function corresponding to an eigenvalue λ_k, then it also contains all the associate functions of higher order corresponding to the same eigenvalue.

Then the SEAF of problem (6), (8) without the set Φ is also complete in the space $L^2[0,1] \oplus L^2[0,1]$.

The proof of this theorem is similar to the proof of Theorem 1.

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