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ON THE INDEX FORMULA FOR SOME NONLOCAL ELLIPTIC BOUNDARY VALUE PROBLEMS

We consider nonlocal elliptic boundary value problems in bounded 2-dimensional domains. In this abstract we describe the case when support of nonlocal terms intersects with boundary of domain. This leads to appearence of power singularities for solutions of nonlocal problems near some points of boundary. Therefore it is natural to consider such problems in weighted Sobolev spaces (see [1, 2]*). The aim of this abstract is to establish a connection between the Fredholm indices of nonlocal problems considered in spaces with different weight indicies.*

Keywords: nonlocal problem, Freholm operator, index of Fredholm operator

Let $G \subset \mathbb{R}^2$ be a bounded domain with the boundary $\partial G = \bar{S}^- \cup \bar{S}^+$, where S^- and S^+ are open smooth curves. The intersection $\bar{S}^- \cap \bar{S}^+$ consists of two points g_1 and g_2.

Let ω be a nondegenerate smooth map of some neighbourhood $\mathcal{O}$ of the curve S^- onto the set $\omega(\mathcal{O})$ such that $\omega(S^-) \subset G$. In this abstract we assume that $\omega(g_i) = g_i$ $(i = 1,\ 2)$.

Suppose that there exists a nondegenerate smooth coordinate transformation $y \mapsto y' = y'(g_i)$ mapping some neighbourhood $\mathcal{V}(g_i)$ of the point g_i onto a neighbourhood of zero $\mathcal{V}(0)$ such that a) the image of the set $G \cap \mathcal{V}(g_i)$ is the intersection of the angle $K_i = \{y \in \mathbb{R}^2 :\ r > 0,\ |\varphi| < b_i\}$ with the neighbourhood $\mathcal{V}(0)$ and the image of the set $S^{\pm} \cap \mathcal{V}(x_0)$ is the intersection of angle's side $\gamma_i^{\pm} = \{y \in \mathbb{R}^2 :\ r > 0,\ \varphi = \pm b_i\}$ with the neighbourhood $\mathcal{V}(0)$ (here $r,\ \varphi$ are polar coordinates in $\mathbb{R}^2$);
b) if $y \in \mathcal{V}(g_i)$, then in new coordinates the map $\omega(y)$ is given by $y' \mapsto \mathcal{G}_i y'$, where $\mathcal{G}_i$ is the map $(\varphi,\ r) \mapsto (\varphi + b_i,\ \chi_i r)$, $\chi_i > 0$.

Let us consider the following nonlocal problem:

$$-\triangle u = f_0(y) \quad (y \in G), \tag{1}$$

$$\begin{gathered} u(y)|_{S^-} + \alpha u(\omega(y))|_{S^-} = f_1(y) \quad (y \in S^-), \\ u(y)|_{S^+} = f_2(y) \quad (y \in S^+), \end{gathered} \tag{2}$$

where $\alpha \in \mathbb{R}$. For simplicity we assume that the boundary condition defined on the curve S^+ does not contain any nonlocal terms.

The principal peculiarity of such problems is the following — the support of nonlocal terms intersects with the boundary of domain: $\overline{\omega(S^-)} \cap \partial G = \{g_1,\ g_2\}$. This leads to appearance of power singularities for solutions in a neighbourhood of the points g_i even for infinitely smooth domain's boundary and an infinitely differentiable right hand side $(f_0,\ f_1,\ f_2)$. Therefore such problems should be considered in some special weighted spaces.

Denote by $H_a^l(G)$ the completion of the set $C_0^\infty(\bar{G}\backslash\{g_1,\ g_2\})$ with respect to the norm

$$\|u\|_{H_a^l(G)} = \Big(\sum_{|\beta| \le l} \int_G \rho^{2(a-l+|\beta|)} |D^\beta u|^2\Big)^{1/2}.$$

Here $l \ge 0$ is an integer; $a \in \mathbb{R}$; $\rho = \rho(y) \in C^\infty(\mathbb{R}^2\backslash\{g_1,\ g_2\})$ in some neighbourhood of the set $\{g_1,\ g_2\}$ is equivalent to the distance between the set $\{g_1,\ g_2\}$ and the point $y \in G$, $\rho(y) \ge c > 0$

outside of this neighbourhood. Denote by $H_a^{l-1/2}(S)$ the space of traces on a smooth curve $S \subset \bar{G}$ with the norm

$$\|\psi\|_{H_a^{l-1/2}(S)} = \inf \|u\|_{H_a^l(G)} \ (u \in H_a^l(G) : \ u|_S = \psi).$$

It was V.A. Kondrat'ev [3] who first considered the weighted spaces $H_a^l(G)$ and $H_a^{l-1/2}(S)$. Constructive descriptions of the trace spaces $H_a^{l-1/2}(S)$ can be also found in [4, §1].

Introduce the bounded operator $\mathcal{L}_a : H_a^2(G) \to H_a^0(G) \times H_a^{3/2}(S^-) \times H_a^{3/2}(S^+)$ given by

$$\mathcal{L}_a u = (-\Delta u(y), \ u(y)|_{S^-} + \alpha u(\omega(y))|_{S^-}, \ u(y)|_{S^+}). \tag{3}$$

The operator $\mathcal{L}_a$ corresponds to nonlocal problem (1), (2).

Our aim is to establish when the operator $\mathcal{L}_a$ is Fredholm and what is a connection between the Fredholm indices of the operator $\mathcal{L}_a$ considered in the weighted spaces with different weight indicies a. In order to answer the formulated questions we write the model problems in the angles K_i, $i = 1, 2$ (see [2]). For this purpose we suppose that $\operatorname{supp} u \subset \mathcal{V}(g_i) \cap \bar{G}$. Denote $U(y') = u(y(y'))$ and replace y' by y. Then problem (1), (2) can be written as follows:

$$\sum_{k,j=1}^{2} p_{ikj}(y) U_{y_k y_j}(y) + \sum_{k=1}^{2} p_{ik}(y) U_{y_k}(y) + p_i(y) U(y) = f_0(y) \quad (y \in K_i),$$

$$U(y)|_{\gamma_i^-} + \alpha U(\mathcal{G}_i y)|_{\gamma_i^-} = f_1(y) \quad (y \in \gamma_i^-),$$
$$U(y)|_{\gamma_i^+} = f_2(y) \quad (y \in \gamma_i^+),$$

where p_{ikj}, p_{ik}, p_i are infinitely differentiable functions.

Suppose for simplicity that

$$p_{ikk}(0) = -1, \quad p_{ikj}(0) = 0 \ (k \neq j). \tag{4}$$

If we now freeze the coefficients at the senior terms at zero and remove the junior terms, then taking into account (4) we get the following model problem in the angle K_i :

$$-\Delta U = f_0(y) \quad (y \in K_i), \tag{5}$$

$$\begin{aligned} U(y)|_{\gamma_i^-} + \alpha U(\mathcal{G}_i y)|_{\gamma_i^-} &= f_1(y) \quad (y \in \gamma_i^-), \\ U(y)|_{\gamma_i^+} &= f_2(y) \quad (y \in \gamma_i^+). \end{aligned} \tag{6}$$

If we now put $(f_0, f_1, f_2) = 0$ in (5), (6), write the function U and the Laplace operator Δ in polar coordinates (r, φ), put $r = e^{-t}$ and do the Fourier transform with respect to t, then we obtain the following auxiliary problem with parameter λ:

$$\tilde{U}_{\varphi\varphi} - \lambda^2 \tilde{U} = 0 \quad (|\varphi| < b_i), \tag{7}$$

$$\begin{aligned} \tilde{U}(\varphi)|_{\varphi=-b_i} + \alpha e^{i\lambda \ln \lambda_i} \tilde{U}(\varphi + b_i)|_{\varphi=-b_i} &= 0, \\ \tilde{U}(\varphi)|_{\varphi=b_i} &= 0. \end{aligned} \tag{8}$$

In [5], it is proved that problem (7), (8) has a discrete spectrum and any strip $\{\lambda \in \mathbb{C} : A_1 < \operatorname{Im} \lambda < A_2\}$ contains no more than a finite number of eigenvalues. The following result was obtained by A.L. Skubachevskii (see [5, 2]).

Theorem 1. *Let condition (4) hold. Then the operator $\mathcal{L}_a$ is Fredholm if and only if the line $\operatorname{Im} \lambda = a - 1$ contains no eigenvalues of auxiliary problems (7), (8) for $i = 1, 2$.*

Now let us consider the operator $\mathcal{L}_{a'} : H_{a'}^2(G) \to H_{a'}^0(G) \times H_{a'}^{3/2}(S^-) \times H_{a'}^{3/2}(S^+)$ given by (3). Here a' is, for determinancy, less than a. So the operators $\mathcal{L}_{a'}$ and $\mathcal{L}_a$ correspond to the same nonlocal problem (1), (2), but act in different weighted spaces. If the lines $\operatorname{Im} \lambda = a' - 1$ and $\operatorname{Im} \lambda = a - 1$ contain no eigenvalues of auxiliary problems (7), (8), $i = 1, 2$, then according to Theorem 1 the operators $\mathcal{L}_{a'}$ and $\mathcal{L}_a$ are both Fredholm.

In order to formulate our main result we denote by λ_{in} ($n = 1, \ldots, N_i$) all the eigenvalues of auxiliary problem (7), (8) ($i = 1, 2$) located between the lines $\operatorname{Im}\lambda = a' - 1$ and $\operatorname{Im}\lambda = a - 1$. Let λ_{in} has an algebraic multiplicity $\varkappa_{in}$ (see corresponding definitions in [6]).

Theorem 2. *Let condition (4) hold. Then*

$$\operatorname{ind}\mathcal{L}_a = \operatorname{ind}\mathcal{L}_{a'} + \sum_{i=1}^{2}\sum_{n=1}^{N_i}\varkappa_{in}.$$

From Theorem 2, it follows in particular that $\operatorname{ind}\mathcal{L}_{a'} = \operatorname{ind}\mathcal{L}_a$ if and only if there are no eigenvalues of auxiliary problems (7), (8), $i = 1, 2$, between the lines $\operatorname{Im}\lambda = a' - 1$ and $\operatorname{Im}\lambda = a - 1$.

Remark 8. In this abstract we considered just a model case. In fact, all the results are obtained for arbitrary elliptic differential operators of order $2m$ and general boundary conditions with a finite number of nonlocal terms.

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