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SMALL FLUCTUATIONS OF VISCOUS MAGNETIZABLE FLUID

Let us consider a closed vessel at rest in a uniform gravitational field filled with a homogeneous incompressible fluid and gas, which is also placed in a magnetic field. We assume that the fluid and gas are non-conductive and their ponderomotive interaction with the magnetic field is caused by magnetization of the substances. We neglect any motion of gas.

We shall denote Ω_1 and Ω_2 to be the volumes occupied with the fluid and gas at the equilibrium state, $\Omega_3 = R^3 \setminus \overline{\Omega}$ to be the unbounded volume outside of the vessel Ω. Let Γ be a free surface of the fluid at the equilibrium state; S be the closed surface of the vessel Ω; S_1 and S_2 be the surfaces of contact, respectively, of the fluid and gas with the vessel walls, such that $S = \overline{S_1 \cup S_2}$. We assume Γ and S to be smooth enough surfaces, which cross one another at each point of the boundary $\partial\Gamma$ of the surface Γ at a nonzero dihedral angle.

For simplicity, we assume that magnetic permeability of substance in the volume Ω_3 is constant. The relations between the magnetic induction $\vec{B}$ and magnetic field strength $\vec{H}$ in each of the volumes Ω_k, $k = \overline{1,3}$ are supposed to have the form:

$$\vec{B}^{(k)} = \overset{0}{\mu}_k \vec{H}^{(k)}, \quad \overset{0}{\mu}_k = \overset{0}{\mu}_k (|H^{(k)}|), \quad k = \overline{1,3},$$

where $\overset{0}{\mu}_k$ is the absolute magnetic permeability of the k's substance. The intensity of the magnetic field $\vec{H}^{(k)}(\vec{x})$ is considered to be a smooth enough function in each volume $\overline{\Omega}_k$, $k = \overline{1,3}$. Let us also suppose that $\overset{0}{\mu}_k (H)$, $k = \overline{1,3}$ are smooth functions, which satisfy the conditions

$$m_0 < \overset{0}{\mu}_k (|H^{(k)}|) \pm |\overset{0}{\mu}{}^{(k)}_H| |H^{(k)}| < m^0, \qquad \forall |H^{(k)}| \geq 0, \qquad k = \overline{1,3},$$

where m_0 and m^0 are some positive constants.

Let $\vec{v}(t, \vec{x})$ be the field of fluid velocities, $\zeta(t, \xi^1, \xi^2)$ be the deviation, counted along the normal $\vec{n}$ to Γ (ξ^1 and ξ^2 are the curvilinear coordinates on Γ), of the free surface $\Gamma'(t)$ from the equilibrium state Γ. $\psi(t, \vec{x})$ is the potential of perturbation of magnetic field caused by the motion of the fluid.

In the linear approximation the motion of capillary fluid near the equilibrium state is described by the following system of equations, as well as the boundary and initial conditions [1, 2, 3]:

$$\rho \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \Delta \vec{v} + \vec{f}, \qquad \operatorname{div} \vec{v} = 0, \qquad (\text{in } \Omega_1), \tag{1}$$

$$\frac{\partial \zeta}{\partial t} = v_n (:= \vec{n} \cdot \vec{v}), \quad (\text{on } \Gamma), \qquad v_{\alpha,3} + v_{3,\alpha} = 0, \quad \alpha = 1, 2, \quad (\text{on } \Gamma), \tag{2}$$

$$-p + 2\rho\nu v_{3,3} = \sigma (-\Delta_\Gamma \zeta + a\zeta) + \left\{ B_n \frac{\partial \psi}{\partial \hat{n}} - \vec{B}_\tau \cdot \nabla_\Gamma \psi \right\}_\Gamma, \qquad (\text{on } \Gamma), \tag{3}$$

$$\zeta = 0, \qquad (\text{on } \partial\Gamma), \qquad \vec{v} = 0, \qquad (\text{on } S), \tag{4}$$

$$\operatorname{div} \mu_k \hat{\nabla} \psi^{(k)} = 0, \qquad k = \overline{1,3}, \qquad (\text{in } \Omega_k), \tag{5}$$

$$\{\psi\}_\Gamma = \{H_n\} \zeta, \qquad \left\{ \mu \frac{\partial \psi}{\partial \hat{n}} \right\}_\Gamma = -\left\{ \operatorname{div}_\Gamma \zeta \vec{B}_\tau \right\}_\Gamma, \quad (\text{on } \Gamma), \tag{6}$$

$$\{\psi\}_S = 0, \qquad \left\{\mu\frac{\partial\psi}{\partial\hat{n}}\right\}_S = 0, \qquad (\text{on } S); \tag{7}$$

$$\psi(t,\vec{x}) \to 0, \qquad \text{if} \quad |\vec{x}| \to \infty, \tag{8}$$

$$\vec{v}(0,\vec{x}) = \vec{v}_0(\vec{x}), \qquad \zeta(0,\xi) = \zeta_0(\xi), \tag{9}$$

$$\hat{\nabla}(\cdot) := \left(\nabla + \frac{\mu_H \vec{H}}{\mu H}(\vec{H}\cdot\nabla)\right)(\cdot), \qquad \mu(\vec{x}) := \overset{0}{\mu}(\vec{H}(\vec{x})),$$

$$\mu_H(\vec{x}) := \frac{d\overset{0}{\mu}(\vec{H}(\vec{x}))}{dH}, \qquad \frac{\partial(\cdot)}{\partial\hat{n}} := \vec{n}\cdot\hat{\nabla}(\cdot);$$

$$a := \frac{\rho}{\sigma}\vec{g}\cdot\vec{n} - (k_1^2 + k_2^2) + \frac{1}{\sigma}\left\{\sum_{\alpha,\beta=1}^{2} b^{\alpha\beta}H_\alpha H_\beta - (k_1+k_2)H_n B_n\right\}_\Gamma .$$

Here ρ and ν are the density and coefficient of kinematic viscosity of the fluid; σ is the coefficient of surface tension on Γ; H_n and $\vec{H}_\tau$ are the projection on the normal and the tangent component of the magnetic field strength at Γ; k_1, k_2 and $b^{\alpha\beta}$ are, respectively, the main curvatures and the components of the second fundamental form of the surface Γ.

For simplicity, we assume that the fluid completely moistens the vessel walls, so that $\partial\Omega_1 = \Gamma\bigcup S$ and $\Gamma\bigcap S = \emptyset$. We set a Hilbert space $\mathcal{H} := \mathcal{L}_2(\Gamma)\ominus\{1\}$. In the Hilbert space $\mathcal{H}$ we shall define the operators

$$\mathcal{L}\zeta := \sigma(-\triangle_\Gamma + a(\vec{x}))\zeta, \quad \mathcal{B}_0\zeta := \mathcal{P}_\mathcal{H}\mathcal{L}\zeta = \mathcal{L}\zeta(\operatorname{mes}\Gamma)^{-1}\int_\Gamma(\mathcal{L}\zeta)d\Gamma, \qquad D(\mathcal{B}_0) := \mathcal{H}^2(\Gamma)\cap\mathcal{H},$$

where $\mathcal{P}_\mathcal{H}$ is the operator of orthogonal projection on the subspace $\mathcal{H}$ in $\mathcal{L}_2(\Gamma)$. One can prove that $\mathcal{B}_0$ is a bounded operator from $\mathcal{H}^{s-1/2}(\Gamma)$ to $\mathcal{H}^{3/2-s}(\Gamma)$, its low bound depends on physical parameters of the system [4].

We note in passing that the trace operator $\gamma^{(k)}$ and operator $\hat{\gamma}^{(k)}\psi^{(k)} := \vec{n}\cdot\psi^{(k)}\big|_\Gamma$, $k = \overline{1,3}$ are bounded operators from $\mathcal{H}^s(\Omega_k)$ to $\mathcal{H}^{s-1/2}(\Gamma)$ and $\mathcal{H}^{s-3/2}(\Gamma)$, respectively, [4]. Under the made assumptions the equation (5) is of uniform elliptical type in Ω_k, $k = \overline{1,3}$. For an arbitrary function $\zeta \in \mathcal{H}^{s-1/2}(\Gamma)$ there exists a unique solution, $\psi = \mathcal{M}\zeta$ ($\mathcal{M}$ is the resolving operator), of the boundary-value problem (5)–(8). For what follows we define the operator

$$\mathcal{B}_1\zeta := \mathcal{P}_\mathcal{H}\left\{B_n(\hat{\gamma}_n\mathcal{M}\zeta) - \vec{B}_\tau\cdot\nabla_\Gamma(\gamma\mathcal{M}\zeta)\right\}_\Gamma = \left\{B_n\frac{\partial\psi}{\partial\hat{n}} - \vec{B}_\tau\cdot\nabla_\Gamma\psi\right\}_\Gamma - (\operatorname{mes}\Gamma)^{-1}\int_\Gamma\left\{B_n\frac{\partial\psi}{\partial\hat{n}} - \vec{B}_\tau\cdot\nabla_\Gamma\psi\right\}_\Gamma. \tag{10}$$

It is easy to show that $\mathcal{B}_1$ is a bounded operator from $\mathcal{H}^{s-1/2}(\Gamma)$ to $\mathcal{H}^{s-3/2}(\Gamma)$. Also is possible to prove that the operator $\mathcal{B}_1 : \mathcal{H}^{3/2}(\Gamma)\cap\mathcal{H} \to \mathcal{H}$ is an unbounded symmetric one.

Let us define the potential energy operator, $\mathcal{B}$, of the system by the relations

$$\mathcal{B} := \mathcal{B}_0 + \mathcal{B}_1, \qquad D(\mathcal{B}) := D(\mathcal{B}_0) = \mathcal{H}^2(\Gamma)\cap\mathcal{H}. \tag{11}$$

Using the Eringen-Nirenberg inequality [5], one can show that $\mathcal{B}$ is a self-adjoined semi-bounded operator in $\mathcal{H}$. In the remaining part of our contribution we shall also use the standard "hydrodynamical" operators A and T [4]. Note that there exists a decomposition $\mathcal{J}_{0,S}(\Omega_1) = \mathcal{M}_0(\Omega_1)\oplus\mathcal{N}_0(\Omega_1)$, where $\mathcal{N}_0(\Omega_1) = A^{1/2}\mathcal{N}_1(\Omega_1)$, $\mathcal{N}^1(\Omega_1) = \{\vec{u}\in\mathcal{J}^1_{0,S}(\Omega_1) : \gamma_n\vec{u} = 0\}$ is the kernel of the operator γ_n [4].

With the help of defined operators the initially-boundary problem (1)–(8) can be presented as a Cauchy problem in the Hilbert space $\mathcal{E} = \mathcal{J}_{0,S}(\Omega_1) \oplus \mathcal{M}_0(\Omega_1)$:

$$\frac{dy(t)}{dt} + \mathcal{A}y(t) = f(t), \tag{12}$$

$$y(0) = y_0, \qquad y_0 \in \mathcal{E}. \tag{13}$$

The operator $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{pmatrix} \nu A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\nu^{-1}Q^*\mathcal{B}Q & -\nu^{-1}Q^*\mathcal{B}Q \\ \nu^{-1}Q^*\mathcal{B}Q & \nu^{-1}Q^*\mathcal{B}Q \end{pmatrix}, \tag{14}$$

where $Q = \gamma_n A^{-1/2}$, $Q = A^{1/2}T$, $y(t) = (\xi(t), \eta(t))^t \in \mathcal{E}$, $y_0 = (\xi(0), \eta(0))^t$ and $f(t) = (A^{1/2}\vec{f}(t); 0)^t$. It turns out that the operators $\mathcal{A}$ and $\mathcal{A}^*$, respectively, have the estimates

$$\operatorname{Re}(\mathcal{A}y, y)_{\mathcal{E}} \geq c\|y\|^2_{\mathcal{E}}, \qquad (y \in D(\mathcal{A})). \tag{15}$$

$$\operatorname{Re}(\mathcal{A}^*y, y)_{\mathcal{E}} \geq c^*\|y\|^2_{\mathcal{E}}, \qquad (y \in D(\mathcal{A}^*)). \tag{16}$$

It follows, that the operator $-\mathcal{A} + cI$ is a maximal dissipative one and, therefore, an estimation holds

$$\|\mathcal{U}(t)\| \leq \exp(-c_m t), \qquad c_m = \max\{c, c^*\}. \tag{17}$$

for a semi-group $\mathcal{U}(t)$ generated by the operator $-\mathcal{A}$ [6].

We then infer that the homogeneous Cauchy problem (12)–(13) is uniformly correct and for its solution $y(t) = \mathcal{U}(t)y_0$ an exponential estimation $\|y(t)\| \leq \exp(-c_m t)\|y_0\|$ holds. If $f(t)$ is a continuous function, which takes values in $\mathcal{E}$ and $y_0 \in \mathcal{E}$, then the non-uniform problem (12)–(13) has the generalized solution $y(t)$ given by the formula

$$y(t) = \mathcal{U}(t)y_0 + \int_0^t \mathcal{U}(t-\tau)f(\tau)d\tau. \tag{18}$$

If $f(t)$ is a continuously differentiable function, which takes values in $\mathcal{E}$, and $y_0 \in \mathcal{A}$, then the non-homogeneous problem (12)–(13) has the classical solution $y(t)$ which is defined by the formula (18).

Let the following conditions, $\vec{v}^0 \in \mathcal{J}^1_{0,S}(\Omega_1)$, $\zeta^0 \in \mathcal{H}^{3/2}_\Gamma(\Gamma)$ and $\vec{f}(t, x)$ is a continuous function of t with values in $\mathcal{J}^1_{0,S}(\Omega_1)$, hold. Then, the initial-boundary problem (1)–(8) has a unique generalized solution, which is continuous in t, with values in $\mathcal{J}^1_{0,S}(\Omega_1)$.

For $f \equiv 0$ the solution of the problem (12)–(13), which depends on t by the law $\exp(-\lambda t)$, describes the normal fluctuations of the system

$$\mathcal{A}y = \lambda y. \tag{19}$$

We draw the reader's attention to the main properties of the spectrum of the problem under investigation. The problem (1)–(8) has a discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$, which is located in the right half-plane with a unique accumulation point $\lambda = \infty$. For any $\varepsilon > 0$ all the eigenvalues λ_j, with exception of, perhaps, their finite number, are located in the sector $|\arg \lambda| < \varepsilon$, i.e. are adjoined to the positive half-axis. The system of the eigen- and adjoined vectors of the problem (19) is complete in the Hilbert space $\mathcal{E}$.

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